

# Structured stability radii and exponential stability tests for Volterra difference systems<sup>☆,☆☆</sup>

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## Abstract

Uniform exponential (UE) stability of linear difference equations with infinite delay is studied using the notions of a stability radius and a phase space. The state space  $\mathcal{X}$  is supposed to be an abstract Banach space. We work both with non-fading phase spaces  $c_0(\mathbb{Z}^-, \mathcal{X})$  and  $\ell^\infty(\mathbb{Z}^-, \mathcal{X})$  and with exponentially fading phase spaces of the  $\ell^p$  and  $c_0$  types. For equations of the convolution type, several criteria of UE stability are obtained in terms of the Z-transform  $\hat{K}(\zeta)$  of the convolution kernel  $K(\cdot)$ , in terms of the input-state operator and of the resolvent (fundamental) matrix. These criteria do not impose additional positivity or compactness assumptions on coefficients  $K(j)$ . Time-varying (non-convolution) difference equations are studied via structured UE stability radii  $r_t$  of convolution equations. These radii correspond to a feedback scheme with delayed output and time-varying disturbances. We also consider stability radii  $r_c$  associated with a time-invariant disturbance operator, unstructured stability radii, and stability radii corresponding to delayed feedback. For all these types of stability radii two-sided estimates are obtained. The estimates from above are given in terms of the Z-transform  $\hat{K}(\zeta)$ , the estimate from below via the norm of the input-output operator. These estimates turn into explicit formulae if the state space  $\mathcal{X}$  is Hilbert or if disturbances are time-invariant. The results on stability radii are applied to obtain various exponential stability tests for non-convolution equations. Several examples are provided.

**Keywords:** discrete Volterra equations, unbounded delay, infinite delay, uniform exponential stability, stability radius, structured perturbations, phase spaces, uncertain feedback, delayed output, delayed feedback

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## 1. Introduction

The aim of the paper is to find or, in more involved cases, to estimate exponential stability radii for linear convolution difference systems with infinite delay

$$x(n+1) = \sum_{j=0}^{+\infty} K(j)x(n-j), \quad n \geq 0, \quad (1.1)$$

and then to apply obtained results to the study of exponential stability of the Volterra difference system

$$x(n+1) = \sum_{j=0}^{+\infty} Q(n,j)x(n-j), \quad n \geq 0, \quad (1.2)$$

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with time-varying (i.e.,  $n$ -depending) coefficients  $Q(n, j)$ . Here  $x(\cdot)$  is a discrete function from  $\mathbb{Z}$  to a (complex) Banach space  $\mathcal{X}$ .  $\mathcal{X}$  is called *the state space*. The coefficients  $Q(n, j)$  belong to the space  $\mathcal{L}(\mathcal{X})$  of bounded linear operators on  $\mathcal{X}$ .

Though Volterra difference systems became an object of active investigations only in last two decades, they have been appearing in various applications at least since 1930s (see the survey [1]). These systems naturally arise in the renewal theory [2], in the numerical studies of Volterra integral equations [3], and in the theory of differential equations with delays [4, 5] (for a list of other applications see [1]). Discretization procedures similar to that of [6, 4, 5] applied to delayed differential and partial differential equations lead to Volterra difference equations with infinite-dimensional state spaces  $\mathcal{X}$ .

Following [7, 8, 9], we consider equations (1.1) and (1.2) in phase space settings. By  $x_n$  the semi-infinite prehistory sequence  $\{\dots, x(n+m), \dots, x(n-1), x(n)\}_{m \leq 0}$  is denoted. We suppose that the sequence of initial conditions  $x_0 = \{x(n+m)\}_{m=-\infty}^0$  (i.e., the prehistory of the initial time point  $n = 0$ ) belongs to a certain *phase space*  $\mathcal{B}$ . In this paper,  $\mathcal{B}$  is either one of exponentially weighted  $\ell^p$ -spaces  $\mathcal{B}^{p,\gamma}$  with the norms

$$|x_0|_{\mathcal{B}^{p,\gamma}} = \left( \sum_{m=-\infty}^0 |e^{\gamma m} x(m)|_{\mathcal{X}}^p \right)^{1/p}, \quad 1 \leq p < \infty, \quad |x_0|_{\mathcal{B}^{\infty,\gamma}} = \sup_{m \leq 0} |e^{\gamma m} x(m)|_{\mathcal{X}}, \quad p = \infty,$$

or one of exponentially weighted  $c_0$ -spaces  $\mathcal{B}_0^{\infty,\gamma}$  with the norms of  $\mathcal{B}^{\infty,\gamma}$  (see the definitions in Section 2.1). Then, all the pre-histories  $x_n$  for  $n \geq 0$  belong to the same phase space  $\mathcal{B}$ . The system

$$x(n+1) = Q(n)x_n, \quad n \geq 0, \quad (1.3)$$

is said to be defined on a phase space  $\mathcal{B}$  if  $Q(n)$  for all  $n \geq 0$  belong to the space  $\mathcal{L}(\mathcal{B}, \mathcal{X})$  of bounded linear operators from  $\mathcal{B}$  to  $\mathcal{X}$ . It is clear that system (1.2) can be written in the form (1.3) if  $Q(n, j)$  satisfy certain assumptions that depend on the choice of the phase space  $\mathcal{B}$ . For instance, (1.2) is defined on  $\mathcal{B}^{p,\gamma}$  whenever  $\sum_{j=0}^{+\infty} \|e^{\gamma j} Q(n, j)\|_{\mathcal{X} \rightarrow \mathcal{X}}^{p'} < \infty$ , where  $p'$  is the Hölder's conjugate of  $p$  (i.e.,  $1/p + 1/p' = 1$ ). This is archived by putting  $Q(n)x_n = \sum_{j=0}^{+\infty} Q(n, j)x(n-j)$ , where the convergence of the series is understood in the sense of the norm topology of  $\mathcal{X}$ . With the same reservations (see also the discussion in Section 3.3), convolution system (1.1) in the phase space settings takes the form of the system  $x(n+1) = Kx_n$  with a time invariant coefficient  $K \in \mathcal{L}(\mathcal{B}, \mathcal{X})$ .

Usually, in the literature the phase spaces  $\mathcal{B}^{\infty,\gamma}$  are used. In [7], such spaces are denoted by  $\mathcal{B}^\gamma$ . When  $\gamma > 0$ , these spaces are called (exponentially) fading because of exponentially decaying term  $e^{\gamma m}$  in the norms. Following the logic of this terminology, it is natural to say that the phase spaces  $\mathcal{B}^{p,\gamma}$  with  $\gamma \leq 0$  are *non-fading*.

In this paper we consider two types of uniform exponential (UE) stability for system (1.3): UE stability in  $\mathcal{X}$  with respect to (w.r.t.) the phase space  $\mathcal{B}$ , and UE stability in the sense of resolvent matrix. The definitions are given in Section 2.2 in accordance with [10, 11, 4, 9]. Note that usually exponential stability for Volterra difference systems (1.2) is understood in the following way.

**Definition 1.1** (see e.g. [12]). *System (1.2) is called exponentially stable if there exist constants  $C, \nu > 0$  such that, for any  $\tau, s \geq 0$ , the solution  $x(n)$  to the problem*

$$\begin{aligned} x(n+1) &= \sum_{j=0}^{n+s-\tau} Q(n, j)x(n-j), \quad n \geq \tau, \\ \{x(\tau-s), \dots, x(\tau-1), x(\tau)\} &= \{\varphi^{[-s]}, \dots, \varphi^{[-1]}, \varphi^{[0]}\} \end{aligned}$$

with arbitrary initial data  $\{\varphi^{[-j]}\}_{j=0}^s \in \mathcal{X}^{s+1}$  satisfies

$$|x(n)|_{\mathcal{X}} \leq C e^{-\nu(n-\tau)} \max_{0 \leq j \leq s} |\varphi^{[-j]}|_{\mathcal{X}} \quad \text{for all } n \geq \tau. \quad (1.4)$$

This type of exponential stability is essentially equivalent to the UE stability in  $\mathcal{X}$  w.r.t.  $\mathcal{B}_0^{\infty,0}$ , see Remark 2.5 (1). That is why we will pay special attention to the phase space  $\mathcal{B}_0^{\infty,0}$  throughout the paper.

We study stability radii associated with the following perturbation of system (1.1)

$$x(n+1) = \sum_{j=0}^{+\infty} K(j) x(n-j) + D N(n) E x_n, \quad (1.5)$$

where  $E \in \mathcal{L}(\mathcal{B}, \mathcal{U}_1)$  and  $D \in \mathcal{L}(\mathcal{U}_2, \mathcal{X})$ . An auxiliary Banach space  $U_2$  ( $U_1$ ) is called the *input* (resp., *output*) *space*. Perturbed system (1.5) can be interpreted as a *feedback system with delayed output*, see Fig.1. Note that the output  $y(n) = E x_n$  depends on the prehistory  $x_n = \{x(n+m)\}_{m=-\infty}^0$  (delayed output) and that the input  $v(n)$  is connected with the output by  $v(n) = N(n)y(n)$ , where  $N(n) \in \mathcal{L}(\mathcal{U}_1, \mathcal{U}_2)$  are operators of uncertain feedback (or disturbance operators).

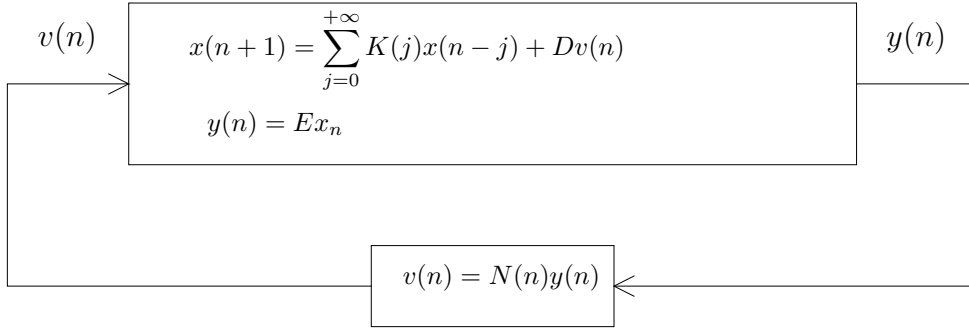


Figure 1: Feedback interpretation of system (1.5)

The (UES) stability radius  $r_t$  is, by definition, a sharp bound on the norms of feedback operators  $N(n)$  that ensures UE stability of the perturbed system (1.5), see Section 4.1 for details. If the feedback operator does not depend on discrete time  $n$ ,  $N(n) \equiv \Delta \in \mathcal{L}(\mathcal{U}_1, \mathcal{U}_2)$ , one gets the stability radius w.r.t. time-invariant structured perturbations  $D\Delta E$ . This radius is denoted  $r_c$ . For systems with bounded delay, more information about structured stability radii and feedback systems can be found in [13, 14] and references therein.

It seems that, for discrete systems with infinite delay, the study of stability radii and of very kindred problems of robust stability was started in the last decade [15, 16, 17, 18, 19]. This theory is not enough developed yet. For convolution system (1.1), asymptotic stability radii corresponding to time-invariant structured perturbations were estimated from above in [18, 19]. These papers assume that the coefficients  $K(j)$  are either positive operators on a finite dimensional state space  $\mathcal{X}$  [19], or are positive compact operators on a complex Banach lattice  $\mathcal{X}$  [18]. Under several additional positivity and compactness assumptions on the perturbations, the stability radius is expressed in terms of the Z-transform  $\hat{K}(1)$  of  $K(\cdot)$  taken at the point  $\zeta = 1$ .

The main points and results of the present paper are:

- Without positivity or compactness assumptions, two-sided estimates for time-varying exponential stability radii  $r_t$  of convolution system (1.1) are obtained (Theorem 4.3, Proposition 4.5, and Theorem 4.6). We work both with the exponentially fading phase spaces and with non-fading phase spaces  $\mathcal{B}_0^{\infty,0}$  and  $\mathcal{B}^{\infty,0}$  (presently, the authors do not know any applications of non-fading phase spaces  $\mathcal{B}^{p,\gamma}$  with  $\gamma < 0$ ). The estimate from above is given in terms of the Z-transform  $\hat{K}(\zeta)$  of the convolution kernel  $K(\cdot)$ , the estimate from below via the norm of the input-output operator  $L_K$ . These results can be seen as an analogue of the stability radii theory for first order systems, see e.g. [14].
- For time-invariant radii  $r_c$ , an explicit formula in terms of Z-transform of  $K(\cdot)$  is given. In the case of a Hilbert state space  $\mathcal{X}$ , we have shown that the same formula is valid for time-varying exponential stability radii  $r_t$  (see formula (4.5) and Theorem 4.6).

- The above mentioned results are used to study unstructured stability radii (Section 5) and stability radii corresponding to a feedback scheme with delayed feedback (Section 7.2).
- As a by-product, in formula (4.5) and Corollary 5.3, we establish connections between the norms of transfer functions and the norms of input-output and unstructured input-state operators. (The authors believe that such formulae can be obtained in a more straight way. Similar results are well known for first order system, see e.g. [14].)
- The results on stability radii are used to obtain various exponential stability tests for time-varying Volterra difference systems (1.2), see Section 7.1. It seems that at least some of these tests are new (related problems were actively discussed e.g. in [11, 20, 9]).

The method used in this paper is a development of that of our previous paper [9] and is based on reduction of system (1.3) to a first order system  $x(n+1) = \mathcal{A}(n)x(n)$ . For the study of stability radii w.r.t. non-fading phase spaces  $\mathcal{B}_0^{\infty,0}$  and  $\mathcal{B}^{\infty,0}$ , we suggest a reduction in two steps: from systems in non-fading phase spaces to system in exponentially fading phase spaces, and then to first order systems (see Sections 6.2 and 6.3). To perform this procedure, we fill two following lacunae in the theories of first order and convolution systems:

- In Section 6.1, we consider first order systems and extend the estimate  $r_t \geq \|L_{\mathcal{A}}\|_{\ell^2(\mathcal{U}_2) \rightarrow \ell^2(\mathcal{U}_1)}^{-1}$  obtained in [14] to  $r_t \geq \|L_{\mathcal{A}}\|_{\ell^q(\mathcal{U}_2) \rightarrow \ell^q(\mathcal{U}_1)}^{-1}$  with arbitrary  $1 \leq q \leq \infty$  (here  $r_t$  and  $L_{\mathcal{A}}$  are the stability radius and the input-output operator corresponding to the first order system, respectively). This extension occurred to be essential for the study of unstructured stability radii for the convolution system and stability radii corresponding to delayed feedback.
- In Section 3.1, criteria of UE stability of system (1.1) are obtained without the assumption of compactness of coefficients  $K(j)$ . The usual assumptions of summability of norms  $\|K(j)\|_{\mathcal{X} \rightarrow \mathcal{X}}$  is also weakened, for details see the discussion in Section 3.3. One of key tools of the proposed reduction method is Theorem 3.3, which shows that the UE stability of convolution system (1.1) w.r.t. the non-fading phase spaces  $\mathcal{B}_0^{\infty,0}$  and  $\mathcal{B}^{\infty,0}$  is equivalent to that w.r.t. fading phase spaces  $\mathcal{B}^{p,\gamma}$  with small positive  $\gamma$ . Continuing the program of [9], we also obtain an exponential stability criterion of Bohl-Perron type for system (1.1) in  $\mathcal{B}^{\infty,0}$ , see Corollary 3.4. It seems that, for time-varying systems in  $\mathcal{B}^{\infty,0}$ , finding of similar criteria is still an open problem.

Another key point of the present paper is the use of phase spaces  $\mathcal{B}^{2,\gamma}$ . The spaces  $\mathcal{B}^{2,\gamma}$  are Hilbert spaces whenever the state space  $\mathcal{X}$  is Hilbert. This fact, in combination with embedding (2.4) of phase spaces, allows us to give an explicit expression for time-varying radii  $r_t$  in formula (4.5), Theorem 4.6, and Corollaries 5.2, 5.3, 7.6.

The paper is organized as follows. After introducing notations and basic stability definitions in Section 2, we present stability results concerning convolution system (1.1) in Section 3. Section 3.3 provides examples to these results and discusses connections with previous studies [12, 4, 5, 25]. In Section 4, perturbed systems are considered: after introducing perturbation types, stability radii, and input-output operators in Section 4.1, the stability radii are estimated in Section 4.2. The proofs of two main results of Section 4, Theorems 4.3 and 4.6, are given in Section 6, which constitutes the main technical part of the paper. Section 5 deals with the important special case of unstructured perturbations. Section 7 presents some applications and examples which illustrate the obtained criteria and estimates: various stability tests for time-varying Volterra difference systems are derived in Section 7.1, stability radii associated with delayed feedback are considered in Section 7.2, and, finally, Section 7.3 provides an example of calculation of stability radii for a non-positive system.

## 2. Notation and basic definitions

We use the convention that the sum equals zero if the lower index exceeds the upper index.

For a set  $S$  in a normed space,  $\overline{S}$  is its closure. The following sets of complex numbers are used:  $\mathbb{T} := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ ,  $\mathbb{D}(\varrho) := \{\zeta \in \mathbb{C} : |\zeta| < \varrho\}$ , and  $\overline{\mathbb{D}(\varrho)} := \{\zeta \in \mathbb{C} : |\zeta| \leq \varrho\}$ . By  $\mathbb{Z}$  and  $\mathbb{Z}^+$  ( $\mathbb{Z}^-$ ), the sets of all integers and all nonnegative (resp., nonpositive) integers are denoted. We write  $\mathbb{Z}_\tau^+$  for the infinite interval of integer numbers in  $[\tau, +\infty)$ . So  $\mathbb{N} = \mathbb{Z}_1^+$ .

### 2.1. Phase spaces, Z-transform, auxiliary operators and functions

Let  $\mathcal{U}, \mathcal{U}_1, \mathcal{U}_2$  be Banach spaces. The norm in  $\mathcal{U}$  is denoted by  $|\cdot|_{\mathcal{U}}$ . Then  $\mathcal{S}(\mathcal{U})$  ( $\mathcal{S}_\pm(\mathcal{U})$ ) denotes the vector space of all *discrete functions*  $v : \mathbb{Z} \rightarrow \mathcal{U}$  (resp.,  $v : \mathbb{Z}^\pm \rightarrow \mathcal{U}$ ). Further,  $\mathcal{L}(\mathcal{U}_1, \mathcal{U}_2)$  denote the Banach space of bounded linear operators from  $\mathcal{U}_1$  to  $\mathcal{U}_2$ ,  $\|\cdot\|_{\mathcal{U}_1 \rightarrow \mathcal{U}_2}$  is the corresponding norm.

The zero vector of a vector space  $\mathcal{W}$  is denoted by  $0_{\mathcal{W}}$ , the identity (zero) operator in  $\mathcal{W}$  by  $I_{\mathcal{W}}$  (resp.,  $\theta_{\mathcal{W}}$ ).

An operator  $G \in \mathcal{L}(\mathcal{U}) := \mathcal{L}(\mathcal{U}, \mathcal{U})$  is called *boundedly invertible* if it is invertible and  $G^{-1} \in \mathcal{L}(\mathcal{U})$ . The *kernel* of an operator  $G \in \mathcal{L}(\mathcal{U})$  is denoted by  $\ker G := \{u \in \mathcal{U} : Gu = 0_{\mathcal{U}}\}$ , and the *image* of  $G$  is

$$\text{range } G := \{u \in \mathcal{U} : u = Gv \text{ for certain } v \in \mathcal{U}\}.$$

The duals of the space  $\mathcal{U}$  and of the operator  $G$  are denoted by  $\mathcal{U}^*$  and  $G^*$ .

We will use the standard Banach spaces  $\ell^p(\mathcal{U}) = \ell^p(\mathbb{Z}^+, \mathcal{U})$  of discrete  $\mathcal{U}$ -valued  $\ell^p$ -functions (so  $\ell^p(\mathcal{U}) \subset \mathcal{S}_+(\mathcal{U})$ ).

The (unilateral) *Z-transform* of a discrete function  $u : \mathbb{Z}^+ \rightarrow \mathcal{U}$  is understood as the power series

$$\widehat{u}(\zeta) = \sum_{j=0}^{+\infty} \zeta^j u(j) \quad (2.1)$$

and, simultaneously, as the corresponding  $\mathcal{U}$ -valued function defined on its set of convergence in  $\mathbb{C}$ , see e.g. [21]. This definition is common for Geophysics. In some papers, the Z-transform of  $u$  is defined as  $\widehat{u}(\zeta^{-1})$ , see e.g. [12]. These definitions are equivalent, but the first is more convenient for us since, if *the corresponding convergence radius*

$$R[\widehat{u}] := \left( \limsup_{j \rightarrow +\infty} |u(j)|_{\mathcal{U}}^{1/j} \right)^{-1} \quad (2.2)$$

is positive, then  $\widehat{u}$  is analytic in the open disc  $\mathbb{D}(R[\widehat{u}])$ .

A function  $u : \mathbb{Z}^+ \rightarrow \mathcal{U}$  is said to *decay exponentially* if  $|u(j)|_{\mathcal{U}} \leq Ce^{-\gamma j}$  for some  $\gamma, C > 0$  (this is equivalent to  $R[\widehat{u}] > 1$ ).

Let a (nontrivial) Banach space  $\mathcal{X}$  be our state space. By  $\mathcal{X}^{\mathbb{Z}^-}$  we denote the vector space of semi-infinite tuples  $\varphi = \{\varphi^{[m]}\}_{m=-\infty}^0$  with elements  $\varphi^{[m]}$  in  $\mathcal{X}$  and indices  $m$  in  $\mathbb{Z}^-$ . We will say that  $\varphi^{[m]}$  is *the m-th coordinate* of  $\varphi$ . The standard notation where  $\mathcal{S}_-(\mathcal{X})$  is used instead of  $\mathcal{X}^{\mathbb{Z}^-}$ , see e.g. [7], is inconvenient in the context of the reduction method used in the present paper.

For  $m \in \mathbb{Z}^-$  we define *the coordinate operator*  $P_m : \mathcal{X}^{\mathbb{Z}^-} \rightarrow \mathcal{X}$  by  $P_m \varphi = \varphi^{[m]}$ . The operator-valued matrix corresponding to  $P_m$  is the row matrix  $(\dots, \theta_{\mathcal{X}}, I_{\mathcal{X}}, \theta_{\mathcal{X}}, \dots, \theta_{\mathcal{X}})$  with the only non-zero entry at  $m$ -th position. The transpose column matrix defines the operator  $P_m^T : \mathcal{X} \rightarrow \mathcal{X}^{\mathbb{Z}^-}$ , i.e.,

$$(P_m^T \psi)^{[j]} = \begin{cases} \psi, & j = m \\ 0_{\mathcal{X}}, & j \neq m \end{cases}, \quad \psi \in \mathcal{X}. \quad (2.3)$$

A linear subspace  $\mathcal{B}$  of  $\mathcal{X}^{\mathbb{Z}^-}$  satisfying a certain set of axioms is called a *phase space* (see e.g. [7]). We will not discuss those axioms since we consider only exponentially weighted  $\ell^p$ - and  $c_0$ -type phase spaces:

$$\begin{aligned} \mathcal{B}^{p,\gamma} &:= \left\{ \{\varphi^{[m]}\}_{m=-\infty}^0 \in \mathcal{X}^{\mathbb{Z}^-} : |\varphi|_{\mathcal{B}^{p,\gamma}} := \left( \sum_{m=-\infty}^0 |e^{\gamma m} \varphi^{[m]}|_{\mathcal{X}}^p \right)^{1/p} < \infty \right\}, \quad 1 \leq p < \infty, \\ \mathcal{B}^{\infty,\gamma} &:= \left\{ \{\varphi^{[m]}\}_{m=-\infty}^0 \in \mathcal{X}^{\mathbb{Z}^-} : |\varphi|_{\mathcal{B}^{\infty,\gamma}} := \sup_{m \in \mathbb{Z}^-} |e^{\gamma m} \varphi^{[m]}|_{\mathcal{X}} < \infty \right\}, \\ \mathcal{B}_0^{\infty,\gamma} &:= \{\varphi \in \mathcal{B}^{\infty,\gamma} : \lim_{m \rightarrow -\infty} |e^{\gamma m} \varphi^{[m]}|_{\mathcal{X}} = 0\}, \quad |\cdot|_{\mathcal{B}_0^{\infty,\gamma}} := |\cdot|_{\mathcal{B}^{\infty,\gamma}}. \end{aligned}$$

In the notation of [7],  $\mathcal{B}^{\infty,\gamma}$  is  $\mathcal{B}^\gamma$ . The spaces  $\mathcal{B}^{p,\gamma}$  with  $p \in [1, \infty)$  were considered in [9]. We will systematically use the fact that if  $\mathcal{X}$  is a Hilbert space, then so are  $\mathcal{B}^{2,\gamma}$ .

The considered class of phase spaces is totally ordered by the continuous embedding:

$$\mathcal{B}^{\infty,\gamma_0} \subset \mathcal{B}^{p_0,\gamma} \subset \mathcal{B}^{p_1,\gamma} \subset \mathcal{B}_0^{\infty,\gamma} \subset \mathcal{B}^{\infty,\gamma} \subset \mathcal{B}^{1,\gamma_1}, \quad 1 \leq p_0 < p_1 < \infty, \quad \gamma_0 < \gamma < \gamma_1. \quad (2.4)$$

For a function  $x(\cdot) \in \mathcal{S}(\mathcal{X})$ ,  $x_n \in \mathcal{X}^{\mathbb{Z}^-}$  denotes the prehistory of  $x(n)$ , i.e.,  $x_n^{[m]} = x(n+m)$ ,  $m \in \mathbb{Z}^-$ . One can see that  $x_n \in \mathcal{B}$  yields  $x_{n+1} \in \mathcal{B}$  for any phase space  $\mathcal{B}$ .

For an operator  $G \in \mathcal{L}(\mathcal{B}, \mathcal{U})$ , let us define a discrete function  $G(\cdot) \in \mathcal{S}_+(\mathcal{L}(\mathcal{X}, \mathcal{U}))$  by

$$G(n) = GP_{-n}^T, \quad n \in \mathbb{Z}^+, \quad \text{and the associated Z-transform by } \widehat{G}(\zeta) = \sum_{j=0}^{+\infty} \zeta^j G(j). \quad (2.5)$$

*Remark 2.1.* The definition of  $G(\cdot)$  is justified by the following: if  $\mathcal{B} = \mathcal{B}^{p,\gamma}$  with  $p \in [1, \infty)$  or  $\mathcal{B} = \mathcal{B}_0^{\infty,\gamma}$ , then  $G\varphi = \sum_{j=0}^{+\infty} G(j)\varphi^{[-j]}$  for all  $\varphi \in \mathcal{B}$ , where the infinite sum is understood in the sense of the strong topology of  $\mathcal{U}$ . When  $p = \infty$ , this representation of  $G$  does not hold for certain  $G \in \mathcal{L}(\mathcal{B}^{\infty,\gamma}, \mathcal{U})$  and  $\varphi \in \mathcal{B}^{\infty,\gamma}$ . Such  $G$  and  $\varphi$  can be constructed, e.g., using Banach limits, see [9, Remark 2.9] (and also Example 3.12 below for another related effect).

**Lemma 2.2.** *Let  $\mathcal{B} = \mathcal{B}_0^{\infty,\gamma}$  or  $\mathcal{B} = \mathcal{B}^{p,\gamma}$  with  $1 \leq p \leq \infty$ . Assume  $G \in \mathcal{L}(\mathcal{B}, \mathcal{U})$ . Then  $\|G(n)\|_{\mathcal{X} \rightarrow \mathcal{U}} \leq e^{-\gamma n} \|G\|_{\mathcal{B} \rightarrow \mathcal{U}}$  for all  $n \in \mathbb{Z}^+$  and  $\gamma$  such that  $e^\gamma \leq R[\widehat{G}]$ .*

*Proof.* The  $\mathcal{B}$ -norm of the tuple  $\varphi = \{\varphi^{[m]}\}_{m=-\infty}^0 = \{\dots, 0_{\mathcal{X}}, 0_{\mathcal{X}}, \psi, 0_{\mathcal{X}}, 0_{\mathcal{X}}, \dots\}$  with the only nonzero entry at  $m_0$ -th position is  $|\varphi|_{\mathcal{B}} = e^{\gamma m_0} |\psi|_{\mathcal{X}}$  (note that  $m_0 \leq 0$ ). Since  $G(-m_0)\psi = GP_{m_0}^T \psi = G\varphi$ , we see that  $\|G(-m_0)\|_{\mathcal{X} \rightarrow \mathcal{U}} \leq e^{\gamma m_0} \|G\|_{\mathcal{B} \rightarrow \mathcal{U}}$  for all  $m_0 \in \mathbb{Z}^-$ . Plugging this into (2.2), one gets  $R[\widehat{G}] \geq e^\gamma$ .  $\square$

## 2.2. Stabilities and the input-state operator

We say that  $Q(\cdot)$  defines the system (1.3) on a phase space  $\mathcal{B}$  if  $Q(n) \in \mathcal{L}(\mathcal{B}, \mathcal{X})$  for all  $n \in \mathbb{Z}^+$ .

From now on assume that  $Q(\cdot)$  defines system (1.3) on a certain phase space  $\mathcal{B}$ . Then, for any  $(\tau, \varphi) \in \mathbb{Z}^+ \times \mathcal{B}$ , there exists unique  $x : \mathbb{Z} \rightarrow \mathcal{X}$  such that  $x_\tau = \varphi$  and (1.3) holds for all  $n \geq \tau$ . The function  $x$  is called a solution to (1.3) through  $(\tau, \varphi)$ , and is denoted by  $x(\cdot, \tau, \varphi)$ . For each  $n \in \mathbb{Z}$ ,  $x_n(\tau, \varphi) := \{x(n+m, \tau, \varphi)\}_{m=-\infty}^0 \in \mathcal{B}$ .

Define the resolvent (fundamental) matrix  $\{X_Q(n, \tau)\}_{n \geq \tau \geq 0}$  by the equalities

$$X_Q(n, \tau)\psi := x(n, \tau, P_0^T \psi), \quad \psi \in \mathcal{X}, \quad (2.6)$$

recall that  $x(\cdot, \tau, P_0^T \psi)$  is the solution to (1.3) satisfying

$$\{\dots, x(\tau-2), x(\tau-1), x(\tau)\} = \{\dots, 0_{\mathcal{X}}, 0_{\mathcal{X}}, \psi\}.$$

So  $X_Q(n, \tau) \in \mathcal{L}(\mathcal{X})$ .

Let us define an unstructured input-state operator  $\Gamma_Q : \mathcal{S}_+(\mathcal{X}) \rightarrow \mathcal{S}_+(\mathcal{X})$  by  $\Gamma_Q(f(\cdot)) = x(\cdot)$ , where  $x = x(\cdot)$  is the solution to the nonhomogeneous system

$$x_0 = 0_{\mathcal{B}}, \quad x(n+1) = \sum_{j=0}^{+\infty} Q(n) x_n + f(n), \quad n \geq 0. \quad (2.7)$$

The unstructured input-state operator and the resolvent matrix are connected by

$$(\Gamma_Q f)(n) = \sum_{j=0}^{n-1} X_Q(n, j+1) f(j) \quad (\text{see e.g. [11]}). \quad (2.8)$$

**Definition 2.3.** *System (1.3) is called uniformly exponentially stable (UES, in short) in (the sense of)  $\mathcal{X}$  with respect to a phase space  $\mathcal{B}$  if it is defined on  $\mathcal{B}$  and there exist constants  $C, \nu > 0$  such that*

$$|x(n, \tau, \varphi)|_{\mathcal{X}} \leq Ce^{-\nu(n-\tau)}|\varphi|_{\mathcal{B}} \quad \text{for all } n, \tau \text{ such that } n \geq \tau \geq 0 \text{ and } \varphi \in \mathcal{B}. \quad (2.9)$$

This stability definition for Volterra systems modifies that of the first order case following the lines of [10] and [9].

In [11], the exponential stability is understood in the resolvent matrix sense.

**Definition 2.4** ([11]). *System (1.3) is called UES in the resolvent matrix sense if there exist  $C, \nu > 0$  such that  $|x(n, \tau, P_0^T \psi)|_{\mathcal{X}} \leq Ce^{-\nu(n-\tau)}|\psi|_{\mathcal{X}}$  for all  $n \geq \tau \geq 0$  and  $\psi \in \mathcal{X}$ , or equivalently,*

$$\|X_Q(n, \tau)\|_{\mathcal{X} \rightarrow \mathcal{X}} \leq Ce^{-\nu(n-\tau)}. \quad (2.10)$$

*Remark 2.5.* (1) The exponential stability introduced by Definition 1.1 is equivalent to the UE stability in  $\mathcal{X}$  w.r.t.  $\mathcal{B}_0^{\infty,0}$  in the following sense. One can define operators  $Q(n)$  on finite tuples  $\varphi = \{\varphi^{[m]}\}_{m=-\infty}^0$  (i.e., on tuples that have a finite number of nonzero entries) by  $Q(n, j)\varphi = \sum Q(n, j)\varphi^{[-j]}$ . Assume that system (1.2) is exponentially stable. Then, operators  $Q(n)$  have a dense in  $\mathcal{B}_0^{\infty,0}$  domain and, by (1.4), are bounded as operators from  $\mathcal{B}_0^{\infty,0}$  to  $\mathcal{X}$ . So they can be extended by continuity to the whole space  $\mathcal{B}_0^{\infty,0}$ . The resulting system (1.3) is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}_0^{\infty,0}$ . Indeed, (1.4) implies (2.9) for finite tuples  $\varphi$ , and passing to limit one can extend (2.9) to all tuples  $\varphi \in \mathcal{B}_0^{\infty,0}$ . Inverting this procedure, one can immediately see that each system (1.3) that is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}_0^{\infty,0}$  produces an exponentially stable system (1.2).

(2) Clearly, for every phase space  $\mathcal{B}$ , the UE stability in  $\mathcal{X}$  w.r.t.  $\mathcal{B}$  implies the UE stability in the resolvent matrix sense.

For  $\mathcal{B} = \mathcal{B}^{p,\gamma}$ , the following criterion of Bohl-Perron type is a reformulation of [9, Theorems 3.1 and 7.2] (see also [22] for  $q_1 = q_2 = \infty$ ). Clearly, the proof of [9, Theorem 3.1] works for  $\mathcal{B} = \mathcal{B}_0^{\infty,\gamma}$  as well.

**Theorem 2.6** ([9]). *Let  $\gamma > 0$ . Let  $\mathcal{B} = \mathcal{B}_0^{\infty,\gamma}$  or  $\mathcal{B} = \mathcal{B}^{p,\gamma}$  with  $1 \leq p \leq \infty$ . Let the (ordered) pair  $(q_1, q_2) \neq (1, \infty)$  be such that  $1 \leq q_1 \leq q_2 \leq \infty$ . Then*

$$(1.3) \text{ is UES in } \mathcal{X} \text{ w.r.t. } \mathcal{B} \Leftrightarrow \Gamma_Q \in \mathcal{L}(\ell^{q_1}(\mathcal{X}), \ell^{q_2}(\mathcal{X})) \text{ and } \sup_{n \in \mathbb{Z}^+} \|Q(n)\|_{\mathcal{B} \rightarrow \mathcal{X}} < \infty. \quad (2.11)$$

The proofs of [9, Theorems 3.1 and 7.2] essentially use the assumption that the phase space is exponentially fading (i.e.,  $\gamma > 0$ ). Among other results of the next section, we give a Bohl-Perron type criterion for Volterra systems of convolution type in the non-fading phase space  $\mathcal{B}_0^{\infty,0}$ .

*Remark 2.7.* Other types of connections between stability and properties of unstructured input input-state operator were considered in [11, 15, 16].

### 3. UE-stability for Volterra systems of convolution type

#### 3.1. Criteria of UE-stability in $\mathcal{B}_0^{\infty,0}$ and in fading phase spaces

Let  $\gamma \in \mathbb{R}$ . Let  $\mathcal{B} = \mathcal{B}_0^{\infty,\gamma}$  or  $\mathcal{B} = \mathcal{B}^{p,\gamma}$  with  $1 \leq p \leq \infty$ . Assume  $K \in \mathcal{L}(\mathcal{B}, \mathcal{X})$  and let  $K(\cdot)$  be the associated discrete function defined by (2.5). In the case  $\mathcal{B} = \mathcal{B}^{\infty,\gamma}$ , we impose the additional technical assumption that

$$K\varphi = \sum_{j=0}^{+\infty} K(j) \varphi^{[-j]} \quad \text{for all } \varphi \in \mathcal{B}^{\infty,\gamma}, \text{ where the infinite sum is understood in the sense}$$

$$\text{of the norm topology of } \mathcal{X}. \quad (3.1)$$

Note that for the other phase spaces  $\mathcal{B}$ , this assumption is always fulfilled due to Remark 2.1.

Recall that  $\hat{K}(\zeta)$  is the Z-transform of the discrete function  $K(\cdot)$ . Lemma 2.2 implies  $R[\hat{K}] \geq e^\gamma$ . That is, the Z-transform  $\hat{K}(\cdot)$  is analytic in  $\mathbb{D}(e^\gamma)$ . The sum  $\sum_{j=0}^{+\infty} K(j) \varphi^{[-j]}$  defines a continuous operator on

$\mathcal{B}^{p,\gamma_1}$  with any  $\gamma_1 < \ln R[\widehat{K}]$  (in particular, with any  $\gamma_1 < \gamma$ ). We keep the same notation  $K$  for all these operators.

In this section, we study the Volterra system of convolution type

$$x(n+1) = \sum_{j=0}^{+\infty} K(j) x(n-j), \quad n \geq 0. \quad (3.2)$$

In our settings, this system can be written in the form  $x(n+1) = Kx_n$  and is defined on the phase space  $\mathcal{B}$  (as well as on the phase spaces  $\mathcal{B}^{p,\gamma_1}$  with  $\gamma_1 < R[\widehat{K}]$ ).

Recall that the unstructured input-state operator  $\Gamma_K$  associated with (3.2) is defined by  $\Gamma_K(f(\cdot)) = x(\cdot)$ , where  $x = x(\cdot)$  is the solution to the nonhomogeneous system

$$x(n+1) = \sum_{j=0}^n K(n-j) x(j) + f(n), \quad n \geq 0, \quad x(0) = 0_{\mathcal{X}}. \quad (3.3)$$

Recall also that an operator  $G \in \mathcal{L}(\mathcal{X})$  is called boundedly invertible if  $\ker G = \{0_{\mathcal{X}}\}$  and  $G^{-1} \in \mathcal{L}(\mathcal{X})$ .

**Theorem 3.1.** *Let  $\gamma > 0$ . Let system (3.2) be defined on  $\mathcal{B}$ , where  $\mathcal{B} = \mathcal{B}_0^{\infty,\gamma}$  or  $\mathcal{B} = \mathcal{B}^{p,\gamma}$  with  $1 \leq p \leq \infty$ . Let*

$$1 \leq q_1 \leq q_2 \leq \infty \quad \text{and} \quad (q_1, q_2) \neq (1, \infty). \quad (3.4)$$

*Then, the following statements are equivalent:*

- (i) *System (3.2) is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}$ .*
- (ii) *For all  $\zeta \in \overline{\mathbb{D}(1)}$ , the operators  $I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)$  are boundedly invertible.*
- (iii) *System (3.2) is UES in the resolvent matrix sense.*
- (iv)  *$\Gamma_K \in \mathcal{L}(\ell^{q_1}, \ell^{q_2})$ .*

The proof is given in Section 3.2. A connection of  $\max_{|\zeta|=1} \|[I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)]^{-1}\|_{\mathcal{X} \rightarrow \mathcal{X}}$  and  $\|\Gamma_K\|_{\ell^{q_1}(\mathcal{X}) \rightarrow \ell^{q_2}(\mathcal{X})}$  is considered in Corollary 5.3. Under certain additional assumptions, the equivalencies (i)  $\Leftrightarrow$  (ii)  $\Leftrightarrow$  (iii) were obtained in [4, Theorems 1 and 2], see for details Remark 3.14 below.

**Proposition 3.2.** *Assume that for another phase space  $\mathcal{B}_1$  the continuous embedding  $\mathcal{B}_1 \subset \mathcal{B}$  holds. Then if system (1.3) is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}$ , it is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}_1$ .*

For the proof, note that the embedding implies that the system is defined on  $\mathcal{B}_1$ . Now the statement follows immediately from the UE stability definition and the continuous embedding inequality  $|\cdot|_{\mathcal{B}} \leq C|\cdot|_{\mathcal{B}_1}$ .

The main result of this section is that, for system (3.2) defined on  $\mathcal{B}_0^{\infty,0}$ , this proposition can be partially reversed.

**Theorem 3.3.** *Let (3.2) be defined on  $\mathcal{B} = \mathcal{B}_0^{\infty,0}$  or on  $\mathcal{B} = \mathcal{B}^{\infty,0}$ . Then, the following statements are equivalent:*

- (i) *System (3.2) is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}$ .*
- (ii) *There exists  $\gamma_0 > 0$  such that (3.2) is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}^{p,\gamma}$  for all  $(p, \gamma) \in [1, \infty] \times (0, \gamma_0]$ .*
- (iii) *System (3.2) is UES in the resolvent matrix sense.*

The proof is given in Section 3.2.

Note the following simple fact:

$$K(\cdot) \text{ decays exponentially} \iff \text{there exists } \gamma_0 > 0 \text{ such that } K \in \mathcal{L}(\mathcal{B}^{p,\gamma}, \mathcal{X}) \text{ for all } (p, \gamma) \in [1, \infty] \times (0, \gamma_0] \quad (3.5)$$

(obviously, 'for all  $(p, \gamma) \in \dots$ ' can be replaced by 'for a certain pair  $(p, \gamma) \in \dots$ ' saving the equivalence). This fact together with Theorems 3.3 and 3.1 implies immediately the following statement, which may be considered as a *Bohl-Perron type criterion for Volterra system of convolution type in  $\mathcal{B}_0^{\infty,0}$* .



**Corollary 3.4.** *Let  $q_1$  and  $q_2$  satisfy (3.4). Then, system (3.2) is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}_0^{\infty,0}$  (w.r.t.  $\mathcal{B}^{\infty,0}$ ) if and only if  $\Gamma_K \in \mathcal{L}(\ell^{q_1}, \ell^{q_2})$  and  $K(\cdot)$  decays exponentially.*

**Corollary 3.5.** *System (3.2) is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}_0^{\infty,0}$  (w.r.t.  $\mathcal{B}^{\infty,0}$ ) if and only if  $K(\cdot)$  decays exponentially and the operators  $I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)$  are boundedly invertible for all  $\zeta \in \overline{\mathbb{D}(1)}$ .*

In the case  $\mathcal{X} = \mathbb{C}^n$ , Corollary 3.5 was obtained in [12, Theorems 5 and 2]. Note that when  $\mathcal{X}$  is finite-dimensional, the condition that  $I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)$  is boundedly invertible for  $\zeta \in \overline{\mathbb{D}(1)}$  turns into the condition

$$\det[I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)] \neq 0, \quad \zeta \in \overline{\mathbb{D}(1)}$$

of [12, Theorem 2]. In the case when  $\mathcal{X}$  is a Banach space and the operators  $K(j)$  are compact, a statement close to Corollary 3.5 follows from [5, Theorems 4 and 2], see for details Remark 3.15 below.

### 3.2. Proofs of Theorems 3.1 and 3.3

Let  $S_{\text{forw}}$  be the right shift in  $\mathcal{S}_+(\mathcal{X})$ , i.e.,  $(S_{\text{forw}}x)(j) = \begin{cases} 0, & j = 0 \\ x(j-1), & j \in \mathbb{N} \end{cases}$ . By  $S_{\text{forw}}^T$  we define the operator with the transpose  $\mathcal{L}(\mathcal{X})$ -valued matrix, i.e.,

$$(S_{\text{forw}}^T x)(j) = x(j+1) \quad \text{for all } j \in \mathbb{Z}^+.$$

In other words,  $S_{\text{forw}}^T$  is the backward shift with truncation of the coordinate with the negative index  $-1$ .

Obviously, for arbitrary  $K(\cdot) \in \mathcal{S}_+(\mathcal{L}(X))$ ,

$$S_{\text{forw}}^T \Gamma_K \text{ is a self-bijection of } \mathcal{S}_+(\mathcal{X}) \quad (3.6)$$

(here  $\Gamma_K$  is the unstructured input-state operator defined via (3.3)).

For convolution system (3.2) the resolvent matrix is a Toeplitz matrix, i.e.,  $X_K(n, j) = X_K(n-j)$  with  $X_K(\cdot) \in \mathcal{S}_+(\mathcal{L}(\mathcal{X}))$ . In particular, one can define the Z-transform  $\widehat{X}_K(\zeta)$  of  $X_K(\cdot)$  (at least as a formal power series).

In the next lemma, assertions (ii) and (iii) are understood in the power series sense,  $(S_{\text{forw}}^T x)^\wedge(\zeta)$  is the Z-transform of  $(S_{\text{forw}}^T x)(\cdot)$  defined by (2.1).

**Lemma 3.6.** *For systems of convolution type, the following assertions are equivalent:*

- (i)  $x(n) = (\Gamma_K f)(n)$ ,  $n \geq 0$ ,
- (ii)  $[I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)] (S_{\text{forw}}^T x)^\wedge(\zeta) = \widehat{f}(\zeta)$  and  $x(0) = 0_{\mathcal{X}}$ ,
- (iii)  $(S_{\text{forw}}^T x)^\wedge(\zeta) = \widehat{X}_K(\zeta) \widehat{f}(\zeta)$  and  $x(0) = 0_{\mathcal{X}}$ .

*Proof.* (i)  $\Leftrightarrow$  (ii). System (3.3) implies  $(S_{\text{forw}}^T x)(n) = (K * x)(n) + f(n)$ , where  $*$  stands for convolution. Applying the Z-transform and taking into account the fact that  $\widehat{x}(\zeta) = \zeta (S_{\text{forw}}^T x)^\wedge(\zeta)$  for  $x$  such that  $x(0) = 0_{\mathcal{X}}$ , we get  $(S_{\text{forw}}^T x)^\wedge(\zeta) = \zeta \widehat{K}(\zeta) (S_{\text{forw}}^T x)^\wedge(\zeta) + \widehat{f}(\zeta)$ . This yields (ii). Inverting the above calculations, we see that (ii)  $\Rightarrow$  (i).

The equivalence (i)  $\Leftrightarrow$  (iii) follows from (2.8), which, for system (3.3), takes the form

$$(\Gamma_K f)(n) = \sum_{j=0}^{n-1} X_K(n-j-1) f(j). \quad (3.7)$$

□

Put

$$R_{\min} := \min\{R(\widehat{K}), R(\widehat{X}_K)\}.$$

**Lemma 3.7.** (i)  $I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)$  and  $\widehat{X}_K(\zeta)$  are two-sided inverses to each other in the ring of formal power series, i.e.,

$$[I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)] \widehat{X}_K(\zeta) \equiv \widehat{X}_K(\zeta) [I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)] \equiv I_{\mathcal{X}}. \quad (3.8)$$

(ii) For all  $\zeta \in \mathbb{D}(R_{\min})$ , the operator  $I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)$  is boundedly invertible and  $[I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)]^{-1} = \widehat{X}_K(\zeta)$ .

*Proof.* It is enough to prove (i), statement (ii) follows immediately from (i). By (3.6), we can test statements (ii) and (iii) of Lemma 3.6 with arbitrary  $x \in \mathcal{S}_+(\mathcal{X})$  satisfying  $x(0) = 0_{\mathcal{X}}$  or with arbitrary  $f \in \mathcal{S}_+(\mathcal{X})$ .

Testing with  $\{x(0), x(1), x(2), x(3), \dots\} = \{0_{\mathcal{X}}, \psi, 0_{\mathcal{X}}, 0_{\mathcal{X}}, \dots\}$ , where  $\psi \in \mathcal{X}$  is arbitrary, we see that:

- the power series  $(S_{\text{forw}}^T x)^\sim(\zeta)$  has only the zero-order term  $\zeta^0 \psi \equiv \psi$ ,
- $[I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)] \psi \equiv \widehat{f}(\zeta)$ ,
- and  $\psi \equiv \widehat{X}_K(\zeta) \widehat{f}(\zeta)$ .

Combining the two last equalities, one gets  $\psi \equiv \widehat{X}_K(\zeta) [I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)] \psi$ . This implies  $\widehat{X}_K(\zeta) [I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)] \equiv I_{\mathcal{X}}$ .

Testing (ii) and (iii) of Lemma 3.6 with  $\{f(0), f(1), f(2), \dots\} = \{\psi, 0_{\mathcal{X}}, 0_{\mathcal{X}}, \dots\}$ , we see that  $[I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)] \widehat{X}_K(\zeta) \equiv I_{\mathcal{X}}$ .  $\square$

Note that (2.10) implies the following equivalence

$$(3.2) \text{ is UES in the resolvent matrix sense } \Leftrightarrow R[\widehat{X}_K] > 1. \quad (3.9)$$

*Proof of Theorem 3.1.* First recall that  $R[\widehat{K}] \geq e^\gamma > 1$  since (3.2) is defined on  $\mathcal{B}$  with  $\gamma > 0$ .

(i)  $\Rightarrow$  (iii). Plugging  $\varphi = \{\dots, \varphi^{[-2]}, \varphi^{[-1]}, \varphi^{[0]}\} = \{\dots, 0_{\mathcal{X}}, 0_{\mathcal{X}}, \psi\}$  into (2.9), we see that the condition of Definition 2.4 is satisfied.

(iii)  $\Rightarrow$  (ii). By (3.9),  $R[\widehat{X}_K] > 1$ . Thus,  $R_{\min} > 1$  and Lemma 3.7 (ii) completes the proof.

(ii)  $\Rightarrow$  (iii). By [23, Sec. VII.6], the set of  $\zeta \in \mathbb{D}(R[\widehat{K}])$  such that  $I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)$  is boundedly invertible is open, and, moreover,  $[I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)]^{-1}$  is analytic on this set. Since  $R[\widehat{K}] > 1$ , (ii) implies that  $[I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)]^{-1}$  is analytic in  $\mathbb{D}(\varrho)$  with certain  $\varrho > 1$ . This and (3.8) imply  $R[\widehat{X}_K] \geq \varrho > 1$ , and, due to (3.9), statement (iii).

(iii)  $\Rightarrow$  (iv). It is enough to consider the case  $q_1 = q_2 = q$ . By (iii),  $X_K(n-j) = X_K(n, j)$  satisfy (2.10). In particular,  $X_K(\cdot) \in \ell^1(\mathcal{L}(\mathcal{X}))$ . Applying Young's inequality for convolutions (see e.g. [23, Problem VI.11.10]) to (3.7), one obtains  $|(\Gamma_K f)(\cdot)|_{\ell^q} \leq |X_K(\cdot)|_{\ell^1} |f(\cdot)|_{\ell^q}$ .

(iv)  $\Rightarrow$  (i) due to Theorem 2.6.  $\square$

**Lemma 3.8.** Assume that  $\varrho > 1$  and that an  $\mathcal{L}(\mathcal{X})$ -valued function  $G$  is analytic in  $\mathbb{D}(\varrho)$ , boundedly invertible in  $\mathbb{D}(1)$ , and  $\sup_{\zeta \in \mathbb{D}(1)} \|G^{-1}(\zeta)\|_{\mathcal{X} \rightarrow \mathcal{X}} = C < \infty$ . Then,  $G$  is boundedly invertible in a certain open neighborhood of  $\overline{\mathbb{D}(1)}$ .

*Proof.* It follows from the assumptions, that  $C > 0$  and the following inequalities hold

$$|G(\zeta)\psi|_{\mathcal{X}} \geq C^{-1}|\psi|_{\mathcal{X}}, \quad |[G(\zeta)]^* \psi^*|_{\mathcal{X}^*} \geq C^{-1}|\psi^*|_{\mathcal{X}^*} \quad \text{for all } \zeta \in \mathbb{D}(1), \psi \in \mathcal{X}, \psi^* \in \mathcal{X}^*. \quad (3.10)$$

For arbitrary  $\zeta_0 \in \mathbb{T} = \{|z| = 1\}$ , let us take  $\{\zeta_n\} \subset \mathbb{D}(1)$  such that  $\zeta_n \rightarrow \zeta_0$  as  $n \rightarrow \infty$ . Passing to the limit in (3.10) and using the continuity of  $G$ , we get for  $\zeta_0 \in \mathbb{T}$ ,

$$|G(\zeta_0)\psi|_{\mathcal{X}} \geq C^{-1}|\psi|_{\mathcal{X}} \quad \text{and} \quad |[G(\zeta_0)]^* \psi^*|_{\mathcal{X}^*} \geq C^{-1}|\psi^*|_{\mathcal{X}^*}. \quad (3.11)$$

This implies that  $\ker G(\zeta_0) = \{0_{\mathcal{X}}\}$  and  $\ker [G(\zeta_0)]^* = \{0_{\mathcal{X}^*}\}$ . The latter equality also implies  $\overline{\text{range } G(\zeta_0)} = \mathcal{X}$  (see [23, Lemma VI.2.8]). On the other hand, (3.11) yields  $\overline{\text{range } G(\zeta_0)} = \text{range } G(\zeta_0)$  (see [23, Exercise VI.9.15]). Hence  $G(\zeta_0)$  is a self-bijection of  $\mathcal{X}$ . By (3.11),  $[G(\zeta_0)]^{-1}$  is bounded.

Thus,  $G(\zeta)$  is boundedly invertible for all  $\zeta \in \overline{\mathbb{D}(1)}$ . The set of  $\zeta$ , where  $G(\zeta)$  is invertible with a bounded inverse, is open in  $\mathbb{D}(\varrho)$  (see [23, Lemma VII.6.1]). So  $G$  is boundedly invertible on  $\mathbb{D}(R_1)$  with certain  $R_1 \in (1, \varrho)$ .  $\square$

**Proposition 3.9.** *Let (3.2) be defined on  $\mathcal{B}_0^{\infty,0}$  and be UES in the resolvent matrix sense. Then, the discrete function  $K(\cdot)$  decays exponentially.*

*Proof.* By the assumptions,  $R[\widehat{X}_K] > 1$  and  $K \in \mathcal{L}(\mathcal{B}_0^{\infty,0}, \mathcal{X})$ . Hence,  $R[\widehat{K}] \geq 1$ . So for all  $\zeta \in \mathbb{D}(1)$ , (3.8) holds true and

$$\|[\widehat{X}_K(\zeta)]^{-1}\|_{\mathcal{X} \rightarrow \mathcal{X}} \leq 1 + \|\widehat{K}(\zeta)\|_{\mathcal{X} \rightarrow \mathcal{X}}. \quad (3.12)$$

On the other hand, for  $\zeta \in \mathbb{D}(1)$ ,  $\psi \in \mathcal{X}$ , and  $\varphi := \{\zeta^{-m}\psi\}_{m=-\infty}^0$ , one has  $|\varphi|_{\mathcal{B}_0^{\infty,0}} \leq |\psi|_{\mathcal{X}}$  and

$$|\widehat{K}(\zeta)\psi|_{\mathcal{X}} = \left| \sum_{j=0}^{+\infty} \zeta^j K(j)\psi \right|_{\mathcal{X}} = |K\varphi|_{\mathcal{X}} \leq \|K\|_{\mathcal{B}_0^{\infty,0} \rightarrow \mathcal{X}} |\psi|_{\mathcal{X}}.$$

So  $\sup_{\zeta \in \mathbb{D}(1)} \|\widehat{K}(\zeta)\|_{\mathcal{X} \rightarrow \mathcal{X}} \leq \|K\|_{\mathcal{B}_0^{\infty,0} \rightarrow \mathcal{X}} < \infty$ . Due to (3.12),  $\sup_{\zeta \in \mathbb{D}(1)} \|[\widehat{X}_K(\zeta)]^{-1}\|_{\mathcal{X} \rightarrow \mathcal{X}} < \infty$ .

Hence, we can apply Lemma 3.8 to the  $\mathcal{L}(X)$ -valued function  $\widehat{X}_K$ . We see that  $\widehat{X}_K$  is boundedly invertible on  $\mathbb{D}(\varrho)$  with certain  $\varrho > 1$ . By (3.8), the function  $\zeta^{-1} \left( I_{\mathcal{X}} - [\widehat{X}_K(\zeta)]^{-1} \right)$  is an analytic continuation of  $\widehat{K}(\zeta)$  from  $\mathbb{D}(1)$  to  $\mathbb{D}(\varrho)$ . Thus,  $R[\widehat{K}] \geq \varrho > 1$ . In other words,  $K(\cdot)$  decays exponentially.  $\square$

*Proof of Theorem 3.3. (iii)  $\Rightarrow$  (ii).* By (3.9),  $R[\widehat{X}_K] > 1$ . By Proposition 3.9,  $K(\cdot)$  decays exponentially. From (3.5), we see that there exists  $\gamma_0 > 0$  such that (3.2) is defined on  $\mathcal{B}^{p,\gamma}$  for all  $(p, \gamma) \in [1, \infty] \times (0, \gamma_0]$ . So the assumption of Theorem 3.1 is fulfilled and the UE stability in the resolvent matrix sense implies the UE stability in  $\mathcal{X}$  w.r.t.  $\mathcal{B}^{p,\gamma}$  for all  $(p, \gamma) \in [1, \infty] \times (0, \gamma_0]$ .

Proposition 3.2 proves the implication (ii)  $\Rightarrow$  (i). For the implication (i)  $\Rightarrow$  (iii) see the proof of Theorem 3.1 (i)  $\Rightarrow$  (iii).  $\square$

### 3.3. Examples and remarks

*Remark 3.10.* In the case  $\mathcal{X} = \mathbb{C}^n$  the proof of Proposition 3.9 can be simplified and Lemma 3.8 is not needed. The reason is the obvious fact that,

$$\text{for } \mathcal{X} = \mathbb{C}^n, \text{ system (3.2) is defined on } \mathcal{B}_0^{\infty,0} \text{ exactly when } K(\cdot) \in \ell^1(\mathcal{L}(\mathcal{X})). \quad (3.13)$$

The proof of Proposition 3.9 can be simplified in the following way (cf. [12, Theorem 4]):  $K(\cdot) \in \ell^1(\mathcal{L}(\mathcal{X}))$  yields that  $\widehat{K}(\zeta)$  is convergent and uniformly bounded in  $\overline{\mathbb{D}(1)}$ . Under the assumptions of Proposition 3.9, one can see that  $\widehat{X}(\zeta)$  is invertible in  $\overline{\mathbb{D}(1)}$ , and so in an open neighborhood of  $\overline{\mathbb{D}(1)}$ . This means that  $\widehat{K}(\zeta)$  is convergent in an open neighborhood of  $\mathbb{D}(1)$ . Thus,  $K(\cdot)$  decays exponentially.

For infinite-dimensional  $\mathcal{X}$ , the condition that (3.2) is defined on  $\mathcal{B}_0^{\infty,0}$  does not imply  $K(\cdot) \in \ell^1(\mathcal{L}(\mathcal{X}))$ . This is shown by the following example.

*Example 3.11.* Let  $\mathcal{X} = c_0$ , where  $c_0$  is the usual Banach space of all convergent to zero sequences  $a = [a_n]_{n=0}^{+\infty} = [a_{[0]}, a_{[1]}, a_{[2]}, \dots]$  of complex numbers. Define an operator  $K \in \mathcal{L}(\mathcal{B}_0^{\infty,0}, c_0)$  by

$$K\varphi = \left[ (\varphi^{[0]})_{[0]}, (\varphi^{[-1]})_{[1]}, (\varphi^{[-2]})_{[2]}, \dots \right]. \quad (3.14)$$

That is,  $K\varphi = \sum_{m=-\infty}^0 K(-m)\varphi^{[m]}$  with  $K(n) \in \mathcal{L}(c_0)$  defined by  $(K(n)a)_{[k]} = \delta_{nk}a_{[n]}$ . Here  $\delta_{nk}$  is Kronecker's delta, and the infinite sum is understood in the strong topology of  $c_0$ .

So  $K$  is a bounded operator from  $\mathcal{B}_0^{\infty,0}$  to  $c_0$ . System (3.2) is defined on  $\mathcal{B}_0^{\infty,0}$ . On the other hand,  $\|K(n)\|_{c_0 \rightarrow c_0} = 1$  and so  $K(\cdot) \notin \ell^1$ .

The following modification of the last example shows that the convolution system (3.2) can be defined on  $\mathcal{B}^{\infty,0}$  under weaker assumptions on  $K(j)$  than (3.1), and that such wider settings may sometimes be more natural.

*Example 3.12.* Let  $\mathcal{X} = \ell^\infty(\mathbb{Z}^+, \mathbb{C})$  be the Banach space of bounded sequences  $[a_n]_{n=0}^{+\infty}$  of complex numbers. Let  $K \in \mathcal{L}(\mathcal{B}^{\infty,0}, \mathcal{X})$  be defined by (3.14). Consider the corresponding discrete function  $K(\cdot)$  (see (2.5) for the definition). Then  $K(n)$  are defined as in Example 3.11, but the infinite sum  $\sum_{m=-\infty}^0 K(-m)\varphi^{[m]}$  is divergent in the strong topology of  $\mathcal{X}$  whenever  $\varphi \in \mathcal{B}^{\infty,0}$  does not satisfy  $\lim_{j \rightarrow +\infty} (\varphi^{[-j]})_{[j]} = 0$ . However, the representation  $K\varphi = \sum_{m=-\infty}^0 K(-m)\varphi^{[m]}$  still holds true for all  $\varphi \in \mathcal{B}^{\infty,0}$  if the sum is understood in the weak\* topology of  $\mathcal{X}$ . (Concerning the representation  $K\varphi = \sum_{m=-\infty}^0 K(-m)\varphi^{[m]}$  in the weak topology of  $\mathcal{X}$ , we refer a reader to the criterion of weak convergence [24, Theorem 8.1.1], [23, Theorem IV.6.31].)

*Remark 3.13.* Theorem 3.1 remains valid if the additional assumption (3.1) is dropped (i.e., it is valid for systems  $x(n+1) = Kx_n$  defined on  $\mathcal{B}^{\infty,\gamma}$ ). Indeed, let the system  $x(n+1) = Kx_n$  be defined on  $\mathcal{B}^{\infty,\gamma}$  with  $\gamma > 0$ . Then (3.1) holds for every  $\varphi \in B^{p,\gamma_1}$  with  $\gamma_1 \in (0, \gamma)$ . In other words, on the narrower space  $B^{p,\gamma_1}$ , the system  $x(n+1) = Kx_n$  takes the convolution form (3.2). Therefore the equivalencies (ii)  $\Leftrightarrow$  (iii)  $\Leftrightarrow$  (iv) of Theorem 3.1 hold true. By Theorem 2.11, the equivalence (i)  $\Leftrightarrow$  (iv) holds for the system  $x(n+1) = Kx_n$  on the original phase space  $\mathcal{B}^{\infty,\gamma}$ . This completes the proof.

*Remark 3.14.* Under the assumption  $\sum_{j=1}^{+\infty} \|e^{j\gamma} K(j)\|_{\mathcal{X} \rightarrow \mathcal{X}} < \infty$ , system (3.2) was studied in [4] in the settings of the phase space  $\mathcal{B}^{\infty,\gamma}$  with  $\gamma > 0$ . In particular, the implication (ii)  $\Rightarrow$  (iii) and the equivalence (i)  $\Leftrightarrow$  (iii) of Theorem 3.1 were proved. The implication (iii)  $\Rightarrow$  (ii) was proved for the case when all operators  $K(j)$  are compact. This compactness assumption is superfluous.

*Remark 3.15.* Under the assumption that  $K(j)$  are compact operators and  $\sum_{j=1}^{+\infty} \|K(j)\|_{\mathcal{X} \rightarrow \mathcal{X}} < \infty$ , it was shown in [5, Theorem 4] that (3.2) is UES in  $X$  w.r.t.  $\mathcal{B}^{\infty,0}$  exactly when (3.2) is uniformly asymptotically stable (UAS) and  $K(\cdot)$  decays exponentially. The result of [5, Theorem 2] and [25, Theorem 1] extend the criteria of UA stability of [12] to Banach space settings imposing the compactness assumption on  $K(j)$  (see also [18] for related results on positive systems). In addition, [5, Remark 1] and [25, Remark 1] discuss the problem of removing the compactness assumption. While our paper is concerned with UE stability, and so does not directly address the problem of [25, Remark 1], Theorem 3.3 and its proof may shed some light on this problem since they do not require the compactness of operators  $K(j)$  for a very kindred question of UE stability.

The following example shows that the condition that system (3.2) is defined on  $\mathcal{B}_0^{\infty,0}$  or  $\mathcal{B}^{p,\gamma}$  with  $\gamma > 0$  cannot be dropped in Theorems 3.1 and 3.3. It also shows that the condition that  $K(\cdot)$  decays exponentially cannot be omitted in Corollary 3.4.

*Example 3.16.* Take  $\mathcal{X} = \mathbb{C}$  and  $K(j) = -2^{j+1}$ . This leads to the system

$$x(n+1) = - \sum_{j=0}^{\infty} 2^{j+1} x(n-j), \quad (3.15)$$

which has the following properties.

- (i)  $R[\widehat{K}] = 1/2$  and system (3.15) is not defined in  $\mathcal{B}^{p,\gamma}$  whenever  $\gamma > -\ln 2$  (for arbitrary  $p$ ). So (3.15) is not UES in  $\mathcal{X}$  w.r.t. these spaces.
- (ii)  $\Gamma_K \in \mathcal{L}(\ell^q)$  for each  $1 \leq q \leq \infty$ .
- (iii) (3.15) is UES in the resolvent matrix sense.

Assertion (i) is obvious. To check (ii) and (iii), note that a solution  $x(\cdot)$  to the nonhomogeneous system  $x(n+1) = - \sum_{j=0}^n 2^{j+1} x(n-j) + f(n)$ ,  $x(0) = 0$ , is given by  $x(1) = f(0)$  and  $x(n) = f(n-1) - 2f(n-2)$ ,  $n \geq 2$ . In particular,  $X_K(1) = -2$ ,  $X_K(n) = 0$  for  $n \geq 2$ .

## 4. Stability radii for various classes of perturbations

### 4.1. Definitions, a feedback scheme with delayed output

Let system (3.2) be defined on a phase space  $\mathcal{B}$ . We consider linear time-invariant and time-varying structured perturbations of (3.2) on  $\mathcal{B}$ . The structure of perturbations is described by the operators  $E \in \mathcal{L}(\mathcal{B}, \mathcal{U}_1)$  and  $D \in \mathcal{L}(\mathcal{U}_2, \mathcal{X})$ , where  $\mathcal{U}_{1,2}$  are auxiliary Banach spaces.

The perturbations of the following types are considered:

$$(\text{Sc}) \quad x(n+1) = \sum_{j=0}^{+\infty} K(j) x(n-j) + D \Delta E x_n,$$

$$(\text{St}) \quad x(n+1) = \sum_{j=0}^{+\infty} K(j) x(n-j) + D \Delta(n) E x_n.$$

The corresponding *disturbance (or unknown feedback) mappings*  $\Delta$  and  $\Delta(n)$  have the following properties:

(Pc)  $\Delta \in \mathcal{L}(\mathcal{U}_1, \mathcal{U}_2)$  is a time-invariant disturbance operator.

(Pt)  $\Delta(\cdot) \in \ell^\infty(\mathcal{L}(\mathcal{U}_1, \mathcal{U}_2))$  is an operator-valued function describing time-varying linear disturbances.

The perturbed systems (Sc)-(St) can be interpreted as *feedback systems with delayed output*, see Fig.1. Note that the output  $y(n) = E x_n$  depends on the prehistory  $x_n = \{x(n+m)\}_{m=-\infty}^0$  (delayed output) and that the input  $v(n)$  is connected with the output by  $v(n) = N(n)y(n)$ , where an unknown operator  $N(n)$  of feedback is given by  $\Delta$  or  $\Delta(n)$ , respectively.  $\mathcal{U}_2$  ( $\mathcal{U}_1$ ) turns into the *input (resp., output)* space.

**Definition 4.1.** The input-output operator  $L_K : \mathcal{S}_+(\mathcal{U}_2) \rightarrow \mathcal{S}_+(\mathcal{U}_1)$  corresponding to Fig.1 is defined by  $L_K : v(\cdot) \rightarrow y(\cdot)$ , where  $y(n) = E \{x(n+m)\}_{m=-\infty}^0$  and  $x(\cdot)$  is the solution to the system

$$x(n+1) = \sum_{j=0}^n K(n-j) x(j) + D v(n), \quad n \geq 0, \quad x_0 = 0_{\mathcal{B}}. \quad (4.1)$$

**Definition 4.2.** The (UE) stability radius of (3.2) w.r.t. perturbations of the structure  $(D, E)$ , the disturbances of the class (Pc), and the phase space  $\mathcal{B}$  is defined by

$$r_c(K; D, E; \mathcal{B}) = \inf\{\|\Delta\|_{\mathcal{U}_1 \rightarrow \mathcal{U}_2} : \Delta \in \mathcal{L}(\mathcal{U}_1, \mathcal{U}_2), \text{ and } (Sc) \text{ is not UES}\}.$$

Usually, we will drop  $K$  in this notation. The stability radius  $r_t(D, E; \mathcal{B})$  w.r.t. the disturbances of the class (Pt) is defined in the analogous way

$$r_c(D, E; \mathcal{B}) = \inf\{|\Delta(\cdot)|_{\ell^\infty} : \Delta(\cdot) \in \ell^\infty(\mathcal{L}(\mathcal{U}_1, \mathcal{U}_2)), \text{ and } (St) \text{ is not UES}\}.$$

Identifying an operator  $\Delta \in \mathcal{L}(\mathcal{U}_1, \mathcal{U}_2)$  with the constant discrete function  $\{\Delta, \Delta, \dots\}$ , one gets a norm-preserving embedding  $\mathcal{L}(\mathcal{U}_1, \mathcal{U}_2) \subset \ell^\infty(\mathcal{L}(\mathcal{U}_1, \mathcal{U}_2))$ . This implies

$$r_c(D, E; \mathcal{B}) \geq r_t(D, E; \mathcal{B}). \quad (4.2)$$

It follows from Proposition 3.2 that, for phase spaces  $\mathcal{B}$  and  $\mathcal{B}_1$ ,

$$\text{continuous embedding } \mathcal{B}_1 \subset \mathcal{B} \implies r_i(D, E; \mathcal{B}_1) \geq r_i(D, E; \mathcal{B}), \quad \text{where } i = c, t. \quad (4.3)$$

4.2. *Main results: stability radii in  $\mathcal{B}_0^{\infty,0}$  and in  $\mathcal{B}^{p,\gamma}$  with  $\gamma > 0$*

By the operator  $E \in \mathcal{L}(\mathcal{B}, \mathcal{U}_1)$ , we define a function  $E(\cdot)$  and the associated Z-transform  $\widehat{E}(\cdot)$  in the way shown by (2.5).

**Theorem 4.3.** Let  $\gamma > 0$  and  $1 \leq q \leq \infty$ . Let  $\mathcal{B} = \mathcal{B}_0^{\infty,\gamma}$  or  $\mathcal{B} = \mathcal{B}^{p,\gamma}$  with  $1 \leq p \leq \infty$ . Let (3.2) be UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}$ . Then  $L_K \in \mathcal{L}(\ell^q(\mathcal{U}_2), \ell^q(\mathcal{U}_1))$  and

$$\left( \max_{|\zeta|=1} \|\widehat{E}(\zeta)[I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)]^{-1} D\|_{\mathcal{U}_2 \rightarrow \mathcal{U}_1} \right)^{-1} = r_c(D, E; \mathcal{B}) \geq r_t(D, E; \mathcal{B}) \geq \|L_K\|_{\ell^q(\mathcal{U}_2) \rightarrow \ell^q(\mathcal{U}_1)}^{-1}. \quad (4.4)$$

If, additionally,  $q = 2$  and  $\mathcal{X}, \mathcal{U}_1, \mathcal{U}_2$  are Hilbert spaces, then (4.4) holds with equalities, i.e.,

$$\left( \max_{|\zeta|=1} \|\widehat{E}(\zeta)[I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)]^{-1} D\|_{\mathcal{U}_2 \rightarrow \mathcal{U}_1} \right)^{-1} = r_c(D, E; \mathcal{B}) = r_t(D, E; \mathcal{B}) = \|L_K\|_{\ell^2(\mathcal{U}_2) \rightarrow \ell^2(\mathcal{U}_1)}^{-1}. \quad (4.5)$$

*Remark 4.4.* Let  $\mathcal{X}$ ,  $\mathcal{U}_1$ , and  $\mathcal{U}_2$  be Hilbert spaces, but  $p \neq 2$ . Then, the phase space  $\mathcal{B}$  is not a Hilbert space, but, according to the theorem, equalities (4.5) still hold (cf. [14, Corollary 4.5]). The proof of this part of the theorem requires an additional step.

The proof is given in Section 6.2.

Let us turn to stability radii in the non-fading phase space  $\mathcal{B}_0^{\infty,0}$ . If  $D = 0_{\mathcal{U}_2 \rightarrow \mathcal{X}}$  or  $E = 0_{\mathcal{B} \rightarrow \mathcal{U}_1}$ , the answer is trivial and not interesting: all the stability radii are equal to  $\infty$ . In the case  $D \neq 0_{\mathcal{U}_2 \rightarrow \mathcal{X}}$ , it occurs that the stability radii may be positive only if the operator  $E$ , which is initially assumed to be in  $\mathcal{L}(\mathcal{B}_0^{\infty,0}, \mathcal{U}_1)$ , satisfies an additional condition.

**Proposition 4.5.** *Let (3.2) be UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}_0^{\infty,0}$  and  $D \neq 0_{\mathcal{U}_2 \rightarrow \mathcal{X}}$ . If  $r_c(D, E; \mathcal{B}_0^{\infty,0}) > 0$ , then the discrete function  $E(\cdot)$  decays exponentially.*

The proof is not too long and illustrates well the use of Theorem 3.3 (i)  $\Leftrightarrow$  (ii).

*Proof.* Assume  $r_c(D, E; \mathcal{B}_0^{\infty,0}) > 0$ . Then for any  $\Delta_0 \in \mathcal{L}(\mathcal{U}_1, \mathcal{U}_2)$  there exists small  $\varepsilon = \varepsilon(\Delta_0) > 0$  such that the time-invariant system (Sc) with  $\Delta = \varepsilon \Delta_0$  is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}_0^{\infty,0}$ . By Theorem 3.3 (i)  $\Leftrightarrow$  (ii), there exists  $\gamma > 0$  such that both systems (3.2) and (Sc) are defined on  $\mathcal{B}^{\infty,\gamma}$ . Hence,

$$\text{the operator } D\Delta E \text{ can be extended by continuity to } \mathcal{B}^{\infty,\gamma}. \quad (4.6)$$

Assume now that  $E(\cdot)$  does not decay exponentially. Then, there exist an increasing sequence  $n_k$  such that  $\lim_{k \rightarrow \infty} \|e^{\gamma n_k} E(n_k)\|_{\mathcal{X} \rightarrow \mathcal{U}_1} = \infty$ . Choose  $\psi(k) \in \mathcal{X}$  with the properties  $|\psi(k)|_{\mathcal{X}} = 1$  and  $|E(n_k)\psi(k)|_{\mathcal{U}_1} > \frac{1}{2} \|E(n_k)\|_{\mathcal{X} \rightarrow \mathcal{U}_1}$  for all  $k \in \mathbb{N}$ . Consider  $\varphi(k) \in \mathcal{B}_0^{\infty,0}$  defined by

$$\varphi^{[m]}(k) = \begin{cases} e^{\gamma n_k} \psi(k) & \text{if } m = -n_k \\ 0_{\mathcal{X}}, & \text{otherwise} \end{cases}.$$

Then

$$|\varphi(k)|_{\mathcal{B}^{\infty,\gamma}} = 1 \quad \text{for all } k, \quad \text{but} \quad \lim_{k \rightarrow \infty} |E\varphi(k)|_{\mathcal{U}_1} = \infty. \quad (4.7)$$

Indeed,  $|E\varphi(k)|_{\mathcal{U}_1} = \left| \sum_{j=0}^{\infty} E(j)\varphi^{[-j]}(k) \right|_{\mathcal{U}_1} = e^{\gamma n_k} |E(n_k)\psi(k)|_{\mathcal{U}_1}$  and so

$$|E\varphi(k)|_{\mathcal{U}_1} \geq \frac{1}{2} e^{\gamma n_k} \|E(n_k)\|_{\mathcal{X} \rightarrow \mathcal{U}_1} \rightarrow \infty.$$

By (4.7) and the uniform boundedness principle, there exists  $u^* \in \mathcal{U}_1^*$  such that  $|u^*(E\varphi(k))| \rightarrow \infty$ . Let  $u_2 \in \mathcal{U}_2$  be such that  $\psi = Du_2 \neq 0_{\mathcal{X}}$ . Consider an operator  $\Delta_0 \in \mathcal{L}(\mathcal{U}_1, \mathcal{U}_2)$  defined by  $\Delta_0 u_1 = u^*(u_1) u_2$ . Then

$$|D\Delta E\varphi(k)|_{\mathcal{X}} = \varepsilon(\Delta_0) |D\Delta_0 E\varphi(k)|_{\mathcal{X}} = \varepsilon(\Delta_0) |\psi|_{\mathcal{X}} |u^*(E\varphi(k))| \rightarrow \infty.$$

This contradicts (4.6).  $\square$

**Theorem 4.6.** *Let  $1 \leq q \leq \infty$ . Let (3.2) be UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}_0^{\infty,0}$ , and let  $E(\cdot)$  decay exponentially. Then  $L_K \in \mathcal{L}(\ell^q(\mathcal{U}_2), \ell^q(\mathcal{U}_1))$  and*

$$\left( \max_{|\zeta|=1} \|\widehat{E}(\zeta)[I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)]^{-1} D\|_{\mathcal{U}_2 \rightarrow \mathcal{U}_1} \right)^{-1} = r_c(D, E; \mathcal{B}_0^{\infty,0}) \geq r_t(D, E; \mathcal{B}_0^{\infty,0}) \geq \|L_K\|_{\ell^q(\mathcal{U}_2) \rightarrow \ell^q(\mathcal{U}_1)}^{-1}. \quad (4.8)$$

*If, additionally,  $q = 2$  and  $\mathcal{X}, \mathcal{U}_1, \mathcal{U}_2$  are Hilbert spaces, then (4.8) holds with equalities.*

The proof is given in Section 6.3.

The assumption that  $E(\cdot)$  decays exponentially is always satisfied in the important case when  $E$  defines perturbations with bounded delay, i.e., when  $E(n) = 0_{\mathcal{X}}$  for  $n$  large enough.

## 5. Unstructured perturbations and the norm of the input-state operator

For a fixed phase space  $\mathcal{B}$ , consider the perturbed system

$$x(n+1) = \sum_{j=0}^{+\infty} K(j) x(n-j) + N(n)x_n, \quad (5.1)$$

where the restrictions similar to (Pc), (Pt) are imposed on the disturbance mappings  $N(n) \in \mathcal{L}(\mathcal{B}, \mathcal{X})$ ,  $n \in \mathbb{Z}^+$ . That is,  $N(n)$  is supposed to be either time invariant  $N(n) = \Delta$  with  $\Delta \in \mathcal{L}(\mathcal{B}, \mathcal{X})$  or time-varying  $N(n) = \Delta(n)$  with  $\Delta(\cdot) \in \ell^\infty(\mathcal{L}(\mathcal{B}, \mathcal{X}))$ .

The definition of the corresponding stability radii  $r_i(K; \mathcal{B})$  can be given in the way similar to that of Definition 4.2, or, alternatively, one can notice that the perturbed systems under consideration are particular cases of (Sc), (St) with the very simple choice of the perturbation structure

$$\mathcal{U}_1 = \mathcal{B}, \quad E = I_{\mathcal{B}}, \quad \mathcal{U}_2 = \mathcal{X}, \quad D = I_{\mathcal{X}}.$$

So, the unstructured stability radii can be defined by

$$r_i(K; \mathcal{B}) := r_i(K; I_{\mathcal{X}}, I_{\mathcal{B}}; \mathcal{B}), \quad i = c, t.$$

The input-output operator  $L_K$  (see Definition 4.1) turns into the unstructured input to prehistory of state operator, i.e.,  $L_K : v(\cdot) \rightarrow x_\bullet$ , where  $x_n = \{x(n+m)\}_{m=-\infty}^0$  and  $x(\cdot)$  is the solution to the system  $x(n+1) = \sum_{j=0}^n K(n-j) x(j) + v(n)$ ,  $x_0 = 0_{\mathcal{B}}$ .

The discrete function  $E(\cdot)$  for  $E = I_{\mathcal{B}^{p,\gamma}}$  is  $E(n) = P_{-n}^T$ , see (2.5).

Let us start with the unstructured radii in the case of the non-fading phase space  $\mathcal{B} = \mathcal{B}_0^{\infty,0}$ . Since  $\|E(n)\|_{\mathcal{X} \rightarrow \mathcal{B}_0^{\infty,0}} = \|P_{-n}^T\|_{\mathcal{X} \rightarrow \mathcal{B}_0^{\infty,0}} = 1$ , we see that the discrete function  $E(\cdot)$  does not decay exponentially. By Proposition 4.5,  $r_c(K; \mathcal{B}_0^{\infty,\gamma}) = 0$ . Due to (4.2), we get the following.

**Corollary 5.1.**  $r_c(K; \mathcal{B}_0^{\infty,0}) = r_t(K; \mathcal{B}_0^{\infty,0}) = 0$ .

Consider fading-phase spaces, i.e., the case when  $\gamma > 0$  and  $\mathcal{B} = \mathcal{B}^{p,\gamma}$  or  $\mathcal{B} = \mathcal{B}_0^{\infty,\gamma}$ .

**Corollary 5.2.** Let  $\gamma > 0$  and  $1 \leq p \leq \infty$ . Let  $\mathcal{B} = \mathcal{B}^{p,\gamma}$  or  $\mathcal{B} = \mathcal{B}_0^{\infty,\gamma}$  (in the latter case it is assumed that  $p = \infty$ ). Let (3.2) be UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}$ . Then  $\Gamma_K \in \mathcal{L}(\ell^p(\mathcal{X}))$  and

$$(1 - e^{-p\gamma})^{1/p} \left( \max_{|\zeta|=1} \| [I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)]^{-1} \|_{\mathcal{X} \rightarrow \mathcal{X}} \right)^{-1} = r_c(K; \mathcal{B}) \geq r_t(K; \mathcal{B}) \geq (1 - e^{-p\gamma})^{1/p} \|\Gamma_K\|_{\ell^p(\mathcal{X}) \rightarrow \ell^p(\mathcal{X})}^{-1}, \quad (5.2)$$

where  $e^{-p\gamma}$  and  $1/p$  have to be understood as zero when  $p = \infty$ .

If, additionally,  $p = 2$  and  $\mathcal{X}$  is a Hilbert space, then (5.2) holds with equalities.

*Proof.* Since  $E(n) = P_{-n}^T$ , we see that  $(\widehat{E}(\zeta)\psi)^{[m]} = \zeta^{-m} P_{-n}^T \psi$  for  $m \in \mathbb{Z}^-$  and  $\psi \in \mathcal{X}$ . Taking  $\zeta \in \{z \in \mathbb{C} : |z| = 1\}$  and  $v \in \mathcal{X}$ , we have for  $p < \infty$

$$\left| \widehat{E}(\zeta) [I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)]^{-1} Dv \right|_{\mathcal{B}^{p,\gamma}}^p = \sum_{m=-\infty}^0 e^{pm\gamma} \left| [I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)]^{-1} v \right|_{\mathcal{X}}^p = (1 - e^{-p\gamma})^{-1} \left| [I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)]^{-1} v \right|_{\mathcal{X}}^p,$$

and  $\left| \widehat{E}(\zeta) [I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)]^{-1} Dv \right|_{\mathcal{B}_{(0)}^{\infty,\gamma}} = \left| [I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)]^{-1} v \right|_{\mathcal{X}}$  when  $p = \infty$ . Theorem 4.3 gives

$$(1 - e^{-p\gamma})^{1/p} \left( \max_{|\zeta|=1} \| (I_{\mathcal{X}} - \zeta \widehat{K}(\zeta))^{-1} \|_{\mathcal{X} \rightarrow \mathcal{X}} \right)^{-1} = r_c(K; \mathcal{B}) \geq r_t(K; \mathcal{B}) \geq \|L_K\|_{\ell^q(\mathcal{X}) \rightarrow \ell^q(\mathcal{B})}^{-1},$$

where  $q$  can be chosen arbitrary in the range  $1 \leq q \leq \infty$ .

With this extremely simple choice of the structure, the operator  $L_K$  can be expressed through the unstructured input-state operator  $\Gamma_K$ . Indeed,

$$(L_K v)(n) = x_n = \{\dots, x(n-1), x(n)\} = \{\dots, (\Gamma_K v)(n-1), (\Gamma_K v)(n)\}. \quad (5.3)$$

Put  $q = p$ . Then, in the case  $p < \infty$ ,

$$\|L_K\|_{\ell^p(\mathcal{X}) \rightarrow \ell^p(\mathcal{B}^{p,\gamma})} = (1 - e^{-p\gamma})^{-1/p} \|\Gamma_K\|_{\ell^p(\mathcal{X}) \rightarrow \ell^p(\mathcal{X})}. \quad (5.4)$$

In fact, (5.3) implies

$$|L_K v|_{\ell^p(\mathcal{B}^{p,\gamma})}^p = \sum_{n=1}^{+\infty} |x_n|_{\mathcal{B}^{p,\gamma}}^p = \sum_{n=1}^{+\infty} \sum_{j=0}^{n-1} |e^{-j\gamma} x(n-j)|_{\mathcal{X}}^p = (1 - e^{-p\gamma})^{-1} \sum_{k=1}^{+\infty} |(\Gamma_K v)(k)|_{\mathcal{X}}^p.$$

When  $p = \infty$ , we obviously have  $\|L_K\|_{\ell^\infty(\mathcal{X}) \rightarrow \ell^\infty(\mathcal{B}^{\infty,\gamma})} = \|\Gamma_K\|_{\ell^\infty(\mathcal{X}) \rightarrow \ell^\infty(\mathcal{X})}$   $\square$

**Corollary 5.3.** *Assume that  $1 \leq p \leq \infty$ ,  $\Gamma_K \in \mathcal{L}(\ell^p(\mathcal{X}))$ , and  $R[\hat{K}] > 1$ . Then  $I_{\mathcal{X}} - \zeta \hat{K}(\zeta)$  is boundedly invertible for all  $\zeta \in \overline{\mathbb{D}(1)}$  and*

$$\max_{|\zeta|=1} \| [I_{\mathcal{X}} - \zeta \hat{K}(\zeta)]^{-1} \|_{\mathcal{X} \rightarrow \mathcal{X}} = \max_{|\zeta| \leq 1} \| [I_{\mathcal{X}} - \zeta \hat{K}(\zeta)]^{-1} \|_{\mathcal{X} \rightarrow \mathcal{X}} \leq \|\Gamma_K\|_{\ell^p(\mathcal{X}) \rightarrow \ell^p(\mathcal{X})}. \quad (5.5)$$

*If, additionally,  $\mathcal{X}$  is a Hilbert space and  $p = 2$ , then the equality hold in (5.5).*

*Proof.* It follows from  $R[\hat{K}] > 1$ , that (3.2) is defined on  $B^{p,\gamma}$  for certain  $\gamma > 0$ . By Theorem 3.1, the assumption  $\Gamma_K \in \mathcal{L}(\ell^p)$  implies the UE stability of (3.2) in  $\mathcal{X}$  w.r.t.  $B^{p,\gamma}$ , and also implies, the bounded invertibility of  $I_{\mathcal{X}} - \zeta \hat{K}(\zeta)$  for all  $\zeta \in \overline{\mathbb{D}(1)}$ . Now (5.5) follows from Corollary 5.2 and the maximum modulus principle.  $\square$

## 6. Proofs of Theorems 4.3 and 4.6: two reductions

### 6.1. Stability radii for first order systems

First, we consider stability radii for a *linear first order time-varying* system

$$w(n+1) = \mathcal{A}(n)w(n), \quad n \in \mathbb{Z}^+, \quad (6.1)$$

where  $w \in \mathcal{S}_+(\mathcal{W})$ ,  $\mathcal{A}(n) \in \mathcal{L}(\mathcal{W})$  for all  $n$ , and  $\mathcal{W}$  is a certain Banach space.

Let  $\mathcal{U}_1, \mathcal{U}_2$  be auxiliary Banach spaces. Let  $\tilde{E} \in \mathcal{L}(\mathcal{W}, \mathcal{U}_1)$  and  $\tilde{D} \in \mathcal{L}(\mathcal{U}_2, \mathcal{W})$ . Following [14], consider two classes of structured perturbations for (6.1)

$$\textbf{(FOSc)} \quad w(n+1) = \mathcal{A}(n)w(n) + \tilde{D} \Delta \tilde{E}w(n),$$

$$\textbf{(FOSt)} \quad w(n+1) = \mathcal{A}(n)w(n) + \tilde{D} \Delta(n) \tilde{E}w(n),$$

where the disturbance mappings  $\Delta$  and  $\Delta(n)$  have the properties (Pc) and (Pt) of Section 4.1, respectively.

The UE stability for first order systems is defined as usual (i.e., the norm  $|\cdot|_{\mathcal{W}}$  replaces both the norms  $|\cdot|_{\mathcal{X}}$  and  $|\cdot|_{\mathcal{B}}$  in Definition 2.3, see e.g. [26, 27, 9]).

The stability radii of (6.1) w.r.t. perturbations of the structure  $(\tilde{D}, \tilde{E})$  and the disturbances classes (Pc) and (Pt) are defined by

$$\begin{aligned} r_c(\mathcal{A}; \tilde{D}, \tilde{E}) &:= \inf\{\|\Delta\|_{\mathcal{U}_1 \rightarrow \mathcal{U}_2} : \Delta \in \mathcal{L}(\mathcal{U}_1, \mathcal{U}_2), \text{ and (FOSc) is not UES} \}, \\ r_t(\mathcal{A}; \tilde{D}, \tilde{E}) &:= \inf\{|\Delta(\cdot)|_{\ell^\infty} : \Delta(\cdot) \in \ell^\infty(\mathcal{L}(\mathcal{U}_1, \mathcal{U}_2)), \text{ and (FOSt) is not UES} \}. \end{aligned}$$



The unstructured input-state operator  $\Gamma_{\mathcal{A}} : \mathcal{S}_+(\mathcal{W}) \rightarrow \mathcal{S}_+(\mathcal{W})$  associated with (6.1) is defined by  $(\Gamma_{\mathcal{A}}f)(\cdot) = w(\cdot)$ , where  $w = w(\cdot)$  is the solution to the nonhomogeneous system

$$w(n+1) = \mathcal{A}(n)w(n) + f(n), \quad n \geq 0, \quad w(0) = 0_{\mathcal{W}}.$$

The input-output operator  $L_{\mathcal{A}} : \mathcal{S}_+(\mathcal{U}_2) \rightarrow \mathcal{S}_+(\mathcal{U}_1)$  corresponding to (6.1) and the perturbation structure  $(\tilde{D}, \tilde{E})$  is defined analogously to Definition 4.1, i.e.,  $(L_{\mathcal{A}}v)(n) := \tilde{E}w(n)$ , where  $w(\cdot)$  is the solution to  $w(n+1) = \mathcal{A}(n)w(n) + \tilde{D}v(n)$ ,  $w(0) = 0_{\mathcal{W}}$ .

For any  $1 \leq q \leq \infty$ , the following criterion of Bohl-Perron type holds

$$\text{system (6.1) is UES} \iff \Gamma_{\mathcal{A}} \in \mathcal{L}(\ell^q(\mathcal{W})), \quad (6.2)$$

see [26, 28, 27] (and also discussion in [9, Sect. 2.2]) for a stronger version of this result. In particular, UE stability of (6.1) implies  $L_{\mathcal{A}} \in \mathcal{L}(\ell^q(\mathcal{U}_2), \ell^q(\mathcal{U}_1))$ .

**Theorem 6.1.** *Suppose (6.1) is UES and  $1 \leq q \leq \infty$ . Then*

$$r_c(\mathcal{A}; \tilde{D}, \tilde{E}) \geq r_t(\mathcal{A}; \tilde{D}, \tilde{E}) \geq \|L_{\mathcal{A}}\|_{\ell^q(\mathcal{U}_2) \rightarrow \ell^q(\mathcal{U}_1)}^{-1}.$$

In the case  $q = 2$ , this result is known, see [14, Theorem 3.1]. The proof of [14] can be modified to cover  $1 \leq q < \infty$  if one uses [29, Theorem 4.2] instead of [14, Proposition 2.4 (iv)]. However this proof does not work when  $q = \infty$ . The following proof, which includes the case  $q = \infty$ , is based on the Bohl-Perron criterion (6.2).

*The proof of Theorem 6.1.* It is enough to prove the second inequality.

For a function  $\Delta : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{U}_1, \mathcal{U}_2)$ , we define the operator  $\mathcal{M}_{\Delta} : \mathcal{S}_+(\mathcal{U}_1) \rightarrow \mathcal{S}_+(\mathcal{U}_2)$  of multiplication on  $\Delta(\cdot)$  by  $(\mathcal{M}_{\Delta}y)(n) = \Delta(n)y(n)$ . Similarly, by  $\mathcal{M}_{\tilde{D}}(\mathcal{M}_{\tilde{E}})$ , the operator of multiplication on  $\tilde{D}$  (resp.,  $\tilde{E}$ ) in the space  $\mathcal{S}_+(\mathcal{U}_2)$  (resp.,  $\mathcal{S}_+(\mathcal{W})$ ) is denoted,

$$(\mathcal{M}_{\tilde{D}}v)(n) = \tilde{D}v(n), \quad v : \mathbb{Z}^+ \rightarrow \mathcal{U}_2, \quad (\mathcal{M}_{\tilde{E}}w)(n) = \tilde{E}w(n), \quad w : \mathbb{Z}^+ \rightarrow \mathcal{W}.$$

The unstructured input-state operator  $\Gamma_{\mathcal{A}}$  and the input-output operator  $L_{\mathcal{A}}$  are connected by

$$L_{\mathcal{A}} = \mathcal{M}_{\tilde{E}}\Gamma_{\mathcal{A}}\mathcal{M}_{\tilde{D}}. \quad (6.3)$$

Suppose (6.1) is UES. Then  $\Gamma_{\mathcal{A}} \in \mathcal{L}(\ell^q(\mathcal{W}))$  and  $L_{\mathcal{A}} \in \mathcal{L}(\ell^q(\mathcal{U}_2), \ell^q(\mathcal{U}_1))$ . Assume that  $|\Delta(\cdot)|_{\ell^\infty} < \|L_{\mathcal{A}}\|_{\ell^q(\mathcal{U}_2) \rightarrow \ell^q(\mathcal{U}_1)}^{-1}$ . Since  $|\Delta(\cdot)|_{\ell^\infty} = \|\mathcal{M}_{\Delta}\|_{\ell^q(\mathcal{U}_1) \rightarrow \ell^q(\mathcal{U}_2)}$ , we have

$$\|\mathcal{M}_{\Delta}L_{\mathcal{A}}\|_{\ell^q(\mathcal{U}_2) \rightarrow \ell^q(\mathcal{U}_2)} < 1.$$

This allows one to define an operator  $\tilde{\Gamma} \in \mathcal{L}(\ell^q(\mathcal{W}))$  by

$$\tilde{\Gamma} := \Gamma_{\mathcal{A}}\mathcal{M}_{\tilde{D}} \left[ \sum_{j=0}^{+\infty} (\mathcal{M}_{\Delta}L_{\mathcal{A}})^j \right] \mathcal{M}_{\Delta}\mathcal{M}_{\tilde{E}}\Gamma_{\mathcal{A}} + \Gamma_{\mathcal{A}}.$$

This definition implies  $\tilde{\Gamma} = \Gamma_{\mathcal{A}}(\mathcal{M}_{\tilde{D}}\mathcal{M}_{\Delta}\mathcal{M}_{\tilde{E}}\tilde{\Gamma} + I_{\ell^q})$ . So  $w(\cdot) = (\tilde{\Gamma}f)(\cdot)$  is the solution to the system

$$w(n+1) = \mathcal{A}(n)w(n) + \tilde{D} \Delta(n) \tilde{E}w(n) + f(n), \quad n \geq 0, \quad w(0) = 0.$$

In other words,  $\tilde{\Gamma}$  is the unstructured input-state operator of the perturbed system (FOSt). Since  $\tilde{\Gamma}$  is bounded in  $\ell^q(\mathcal{W})$ , the Bohl-Perron criterion (6.2) implies that the perturbed system (FOSt) is UES. This completes the proof.  $\square$

Now we apply the above theorem to strengthen some of the results of [14] on *linear first order time-invariant systems* so that they fit to our needs.

When  $\mathcal{A}(n) = A$  for all  $n$  with  $A \in \mathcal{L}(\mathcal{W})$ , system (6.1) takes the form

$$w(n+1) = Aw(n), \quad n \in \mathbb{Z}^+. \quad (6.4)$$

The corresponding input-output operator and stability radii are denoted by  $L_A$  and  $r_i(A; \tilde{D}, \tilde{E})$ ,  $i = c, t$ , respectively.

**Theorem 6.2** (cf. [14]). *Suppose (6.4) is UES and  $1 \leq q \leq \infty$ . Then  $L_A \in \mathcal{L}(\ell^q(\mathcal{U}_2), \ell^q(\mathcal{U}_1))$  and the following statements hold.*

- (i)  $\left( \max_{|\lambda|=1} \|\tilde{E}(\lambda I_{\mathcal{W}} - A)^{-1} \tilde{D}\|_{\mathcal{U}_2 \rightarrow \mathcal{U}_1} \right)^{-1} = r_c(A; \tilde{D}, \tilde{E}) \geq r_t(A; \tilde{D}, \tilde{E}) \geq \|L_A\|_{\ell^q(\mathcal{U}_2) \rightarrow \ell^q(\mathcal{U}_1)}^{-1}$ .
- (ii) *If, additionally,  $\mathcal{W}, \mathcal{U}_1, \mathcal{U}_2$  are Hilbert spaces (and  $q = 2$ ), then*

$$\left( \max_{|\lambda|=1} \|\tilde{E}(\lambda I_{\mathcal{W}} - A)^{-1} \tilde{D}\|_{\mathcal{U}_2 \rightarrow \mathcal{U}_1} \right)^{-1} = r_c(A; \tilde{D}, \tilde{E}) = r_t(A; \tilde{D}, \tilde{E}) = \|L_A\|_{\ell^2(\mathcal{U}_2) \rightarrow \ell^2(\mathcal{U}_1)}^{-1}.$$

Recall that time-invariant system (6.4) is UES if and only if the spectral radius of  $A$  is less than 1. Statement (ii) and, in the case  $q = 2$ , statement (i) of this theorem follows immediately from a combination of [14, Corollary 4.5 and Proposition 5.3]. Statement (i) for  $q \neq 2$  is a combination of the above mentioned results of [14] with Theorem 6.1.

### 6.2. The proof of Theorem 4.3: reduction of order

Let  $\gamma \in \mathbb{R}$  and let  $\mathcal{B} = \mathcal{B}^{p,\gamma}$  or  $\mathcal{B} = \mathcal{B}_0^{\infty,\gamma}$ . We want to write the Volterra convolution system (3.2) defined on the phase space  $\mathcal{B}$  and the perturbed systems (Sc)–(St) in the form of first order systems.

Recall that system (3.2) can be written in the form  $x(n+1) = Kx_n$ , where  $K \in \mathcal{L}(\mathcal{B}, \mathcal{X})$ . Define the backward shift operator  $S_{B,\text{back}}$  in  $\mathcal{X}^{\mathbb{Z}^-}$  (and so in all the phase spaces) by

$$(S_{B,\text{back}}\varphi)^{[m]} := \begin{cases} 0_{\mathcal{X}}, & m = 0, \\ \varphi^{[m+1]}, & m \leq -1 \end{cases}.$$

Then the first order system (6.4) with

$$A := P_0^T K + S_{B,\text{back}} \in \mathcal{L}(\mathcal{B}) \quad (6.5)$$

and  $\mathcal{W} = \mathcal{B}$  is associated with system (3.2) in the sense that

$$x_n(\tau, \varphi) = w(n, \tau, \varphi), \quad (6.6)$$

where  $w(\cdot, \tau, \psi)$  is a unique solution to system (6.4) satisfying the initial condition  $w(\tau) = \psi$ . The operator  $A$  can be written in the form of matrix with  $\mathcal{L}(\mathcal{X})$ -entries:

$$A = \begin{pmatrix} K(0) & K(1) & K(2) & \dots & K(j-1) & K(j) & \dots \\ I_{\mathcal{X}} & 0_{\mathcal{X}} & 0_{\mathcal{X}} & \dots & 0_{\mathcal{X}} & 0_{\mathcal{X}} & \dots \\ 0_{\mathcal{X}} & I_{\mathcal{X}} & 0_{\mathcal{X}} & \dots & 0_{\mathcal{X}} & 0_{\mathcal{X}} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0_{\mathcal{X}} & 0_{\mathcal{X}} & 0_{\mathcal{X}} & \dots & 0_{\mathcal{X}} & 0_{\mathcal{X}} & \dots \\ 0_{\mathcal{X}} & 0_{\mathcal{X}} & 0_{\mathcal{X}} & \dots & I_{\mathcal{X}} & 0_{\mathcal{X}} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Given the structure  $\{E, D\}$  of perturbations (Sc)–(St), we define the structure  $\{\tilde{E}, \tilde{D}\}$  of perturbations (FOSc)–(FOSnt) putting

$$\tilde{E} := E, \quad \tilde{D} := P_0^T D. \quad (6.7)$$

Then solutions of (Sc)-(St) and of (FOSc)-(FOSt), resp., are also connected by (6.6). Moreover, the input-output operators are identical

$$L_A = L_K. \quad (6.8)$$

The above procedure may be considered as a generalization to systems with infinite delay of the phase-space method, which is well developed for systems with bounded delay, see e.g. [13].

**Proposition 6.3.** *Let  $\gamma > 0$ . Let  $\mathcal{B} = \mathcal{B}^{p,\gamma}$  or  $\mathcal{B} = \mathcal{B}_0^{\infty,\gamma}$ . Assume (6.5) and (6.7). Then:*

- (i) *system (St) is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}$  if and only if (FOSt) is UES,*
- (ii)  $r_i(K; D, E; \mathcal{B}) = r_i(A; \tilde{D}, \tilde{E}), i = c, t.$

Statement (i) is proved in our previous paper [9, Proposition 3.12 and Sec. 7.2], statement (ii) follows from (i).

**Proposition 6.4.** *Assume (6.5) and (6.7). Assume that both  $\zeta^{-1}I_{\mathcal{B}} - A$  and  $I_{\mathcal{X}} - \zeta\hat{K}(\zeta)$  are boundedly invertible for a certain  $\zeta \in \mathbb{C} \setminus \{0\}$ . Then for any  $v \in \mathcal{U}_2$ :*

- (i)  $(\zeta^{-1}I_{\mathcal{B}} - A)^{-1}\tilde{D}v = \left\{ \zeta^{-m+1}[I_{\mathcal{X}} - \zeta\hat{K}(\zeta)]^{-1}Dv \right\}_{m=-\infty}^0,$
- (ii) *if additionally  $\zeta \in \mathbb{D}(e^\gamma) \setminus \{0\}$  (with  $\gamma$  from the definition of  $\mathcal{B}$ ), then*

$$\tilde{E}(\zeta^{-1}I_{\mathcal{B}} - A)^{-1}\tilde{D} = \zeta\hat{E}(\zeta)[I_{\mathcal{X}} - \zeta\hat{K}(\zeta)]^{-1}D.$$

*Proof.* (i) Due to (6.5) and (6.7), the equality  $(\zeta^{-1}I_{\mathcal{B}} - A)\varphi = \tilde{D}v$  can be rewritten as the system

$$\begin{aligned} \zeta^{-1}\varphi^{[0]} &= \sum_{j=0}^{+\infty} K(j)\varphi^{[-j]} = Dv \quad (\text{for } m = 0), \\ \zeta\varphi^{[m+1]} &= \varphi^{[m]} \quad \text{for } m = -1, -2, \dots \end{aligned}$$

This leads to  $\varphi^{[m]} = \zeta^{-m}\varphi^{[0]}, m \leq -1$ , and in turn to  $\varphi^{[0]} = \zeta[I_{\mathcal{X}} - \zeta\hat{K}(\zeta)]^{-1}Dv$ . So  $\varphi$  is found and gives (i).

(ii) *First, take a simplified point of view that*

$$E\varphi = \sum_{j=0}^{+\infty} E(j)\varphi^{[-j]} \quad (6.9)$$

for all  $\varphi \in \mathcal{B}$  (this holds for each  $E \in \mathcal{L}(\mathcal{B}, \mathcal{U}_1)$  in all the phase spaces except  $\mathcal{B}^{\infty,\gamma}$ , see Remark 2.1). Since  $E \in \mathcal{L}(\mathcal{B}, \mathcal{U}_1)$  (where  $\mathcal{B} = \mathcal{B}^{p,\gamma}$  or  $\mathcal{B} = \mathcal{B}_0^{\infty,\gamma}$ ), we see that  $\hat{E}(\zeta)$  is defined for  $|\zeta| < e^\gamma$ . Since  $\tilde{E} = E$ , we get using (i) that

$$\tilde{E}(\zeta^{-1}I_{\mathcal{B}} - A)^{-1}\tilde{D}v = \sum_{j=0}^{+\infty} \zeta^{j+1}E(j)[I_{\mathcal{X}} - \zeta\hat{K}(\zeta)]^{-1}Dv = \zeta\hat{E}(\zeta)[I_{\mathcal{X}} - \zeta\hat{K}(\zeta)]^{-1}Dv, \quad \zeta \in \mathbb{D}(e^\gamma) \setminus \{0\}.$$

Now assume  $\mathcal{B} = \mathcal{B}^{\infty,\gamma}$ . Generally, (6.9) does not hold in  $\mathcal{B} = \mathcal{B}^{\infty,\gamma}$ . However, (i) implies that  $\varphi_0 := (\zeta^{-1}I_{\mathcal{B}} - A)^{-1}\tilde{D}v$  has the form  $\{\zeta^{-m}\varphi_0^{[0]}\}_m$ . So, for  $|\zeta| < e^\gamma$ , we see that  $\varphi_0 \in \mathcal{B}_0^{\infty,\gamma}$ . Since the representation (6.9) holds for all  $\varphi \in \mathcal{B}_0^{\infty,\gamma}$ , it holds for  $\varphi_0$ .  $\square$

*Proof of Theorem 4.3. Step 1: the proof of (4.4).* By Proposition 6.3 (i), system (6.4) is UES exactly when (3.2) is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}$ . The boundness of  $L_K$  follows from those of  $L_A$ , see (6.8) and the remarks before Theorem 6.1. Formula (4.4) follows from Theorem 6.2 (i), Propositions 6.3 (ii) and 6.4 (ii). Note that Proposition 6.4 (ii) is applicable since  $\zeta$  in (4.4) belongs to the unit circle and  $e^\gamma > 1$ .

*Step 2: the proof of (4.5) for the case  $p = 2$ .* Since  $\mathcal{X}$  is a Hilbert space, we see that  $\mathcal{W} = \mathcal{B} = \mathcal{B}^{2,\gamma}$  is so. Theorem 6.2 (ii) and Propositions 6.3-6.4 give

$$r_c(K; D, E; \mathcal{B}) = r_t(K; D, E; \mathcal{B}) = \|L_K\|_{\ell^2(\mathcal{U}_2) \rightarrow \ell^2(\mathcal{U}_1)}^{-1} = \left( \max_{|\zeta|=1} \|\widehat{E}(\zeta)(I_{\mathcal{X}} - \zeta \widehat{K}(\zeta))^{-1} D\|_{\mathcal{U}_2 \rightarrow \mathcal{U}_1} \right)^{-1}.$$

*Step 3: the proof of (4.5) for  $p \neq 2$ .* Let us take  $\gamma_1 \in (0, \gamma)$  and apply formula (4.3) to the continuous embedding  $\mathcal{B}^{2,\gamma_1} \subset \mathcal{B}$ . This gives  $r_c(K; D, E; \mathcal{B}^{2,\gamma_1}) \geq r_c(K; D, E; \mathcal{B})$ . Formula (4.5) for  $\mathcal{B}^{2,\gamma_1}$  has been proved already on Step 2 and gives

$$r_c(K; D, E; \mathcal{B}^{2,\gamma_1}) = \|L_K\|_{\ell^2(\mathcal{U}_2) \rightarrow \ell^2(\mathcal{U}_1)}^{-1} = \left( \max_{|\zeta|=1} \|\widehat{E}(\zeta)[I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)]^{-1} D\|_{\mathcal{U}_2 \rightarrow \mathcal{U}_1} \right)^{-1}.$$

These two formulae lead to

$$\left( \max_{|\zeta|=1} \|\widehat{E}(\zeta)[I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)]^{-1} D\|_{\mathcal{U}_2 \rightarrow \mathcal{U}_1} \right)^{-1} = \|L_K\|_{\ell^2(\mathcal{U}_2) \rightarrow \ell^2(\mathcal{U}_1)}^{-1} \geq r_c(K; D, E; \mathcal{B}).$$

Combining the latter with formula (4.4) obtained on Step 1, we complete the proof.  $\square$

### 6.3. Proof of Theorem 4.6: reduction to exponentially fading phase spaces

Let system (3.2) be UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}_0^{\infty,0}$ . According to Theorem 3.3 (i)  $\Leftrightarrow$  (ii), system (3.2) is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}^{2,\gamma}$  for all  $\gamma \in (0, \gamma_0)$  with certain  $\gamma_0 > 0$ . This and Theorem 4.3 imply  $L_K \in \mathcal{L}(\ell^q(\mathcal{U}_2), \ell^q(\mathcal{U}_1))$ .

From the assumption that  $E(\cdot)$  decays exponentially, we see that there exists  $\gamma_1 > 0$  such that the operator  $E$  can be extended by continuity to the spaces  $\mathcal{B}^{2,\gamma}$  with  $\gamma \in (0, \gamma_1)$ . We keep the same notation  $E$  for these extensions. Put

$$\gamma_2 := \min\{\gamma_0, \gamma_1\} \quad \text{and} \quad r_0 := \left( \max_{|\zeta|=1} \|\widehat{E}(\zeta)[I_{\mathcal{X}} - \zeta \widehat{K}(\zeta)]^{-1} D\|_{\mathcal{U}_2 \rightarrow \mathcal{U}_1} \right)^{-1}.$$

Theorem 4.3 and implication (4.3) yield that for all  $\gamma \in (0, \gamma_2)$

$$r_c(D, E; \mathcal{B}_0^{\infty,0}) \geq r_c(D, E; \mathcal{B}^{2,\gamma}) = r_0, \tag{6.10}$$

$$r_t(D, E; \mathcal{B}_0^{\infty,0}) \geq r_t(D, E; \mathcal{B}^{2,\gamma}) \geq \|L_K\|_{\ell^q(\mathcal{U}_2) \rightarrow \ell^q(\mathcal{U}_1)}^{-1}. \tag{6.11}$$

In particular,  $r_c(D, E; \mathcal{B}^{2,\gamma})$  does not depend on the choice of  $\gamma \in (0, \gamma_2)$ .

Let us prove that

$$r_c(D, E; \mathcal{B}_0^{\infty,0}) = r_0.$$

Taking (6.10) into account, it is enough to prove that  $r_c(D, E; \mathcal{B}_0^{\infty,0}) \leq r_0$ . Assume that the time-invariant system (Sc) is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}_0^{\infty,0}$ . Then, by Theorem 3.3 (i)  $\Leftrightarrow$  (ii) applied to (Sc), system (Sc) is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}^{2,\gamma}$  for certain  $\gamma \in (0, \gamma_2)$ . The definition of  $r_c(D, E; \mathcal{B}^{2,\gamma})$  imply that  $\|\Delta\|_{\mathcal{U}_1 \rightarrow \mathcal{U}_2} \leq r_c(D, E; \mathcal{B}^{2,\gamma}) = r_0$  (see anew (6.10) and (Sc)). This imply the desired statement.

Combining the equality  $r_c(D, E; \mathcal{B}_0^{\infty,0}) = r_0$  with (4.2) and (6.11) we get (4.8).

When  $\mathcal{X}$  and  $\mathcal{U}_{1,2}$  are Hilbert spaces and  $q = 2$ , Theorem 4.3 implies  $r_0 = \|L_K\|_{\ell^2(\mathcal{U}_2) \rightarrow \ell^2(\mathcal{U}_1)}^{-1}$ . From this and (4.8), one can see that (4.8) holds with the equalities.

## 7. Applications to systems of special types and examples

### 7.1. Sufficient conditions for UE stability of time-varying systems

The following lemma is standard and can be proved in the same way as in the first order case.

**Lemma 7.1.** *Assume that  $\mathcal{B}$  is one of the phase spaces considered in Section 2.1. Let  $Q(n) \in \mathcal{L}(\mathcal{B}, \mathcal{X})$  and  $\tilde{Q}(n) \in \mathcal{L}(\mathcal{B}, \mathcal{X})$  for all  $n \in \mathbb{Z}^+$ . If  $Q(n) = \tilde{Q}(n)$  for  $n$  large enough, then the UE stabilities in  $\mathcal{X}$  w.r.t.  $\mathcal{B}$  of the systems  $x(n+1) = Q(n)x_n$  and  $x(n+1) = \tilde{Q}(n)x_n$  are equivalent.*

Roughly speaking, a modification of a finite number of operators  $Q(n)$  in the system  $x(n+1) = Q(n)x_n$  does not influence its UE stability.

Let  $E(j) \in \mathcal{L}(\mathcal{X}, \mathcal{U})$  and  $\Delta(j) \in \mathcal{L}(\mathcal{U}, \mathcal{X})$  for all  $j \in \mathbb{Z}^+$ . Consider the system

$$x(n+1) = \Delta(n) \sum_{j=0}^{+\infty} E(j)x(n-j), \quad n \geq 0. \quad (7.1)$$

Let us apply Theorem 4.3 to system (7.1).

**Corollary 7.2.** *Let  $\mathcal{X}$  and  $\mathcal{U}$  be Hilbert spaces and  $\gamma > 0$ . Let  $\|E(j)\|_{\mathcal{X} \rightarrow \mathcal{U}} \leq Ce^{-\gamma j}$  for all  $j \in \mathbb{Z}^+$  with a certain constant  $C$ . Then system (7.1) is UES in  $\mathcal{X}$  w.r.t.  $B^{1,\gamma}$  (and so w.r.t. all  $B^{p,\beta}$  with  $\beta < \gamma$ ) whenever*

$$\limsup_{n \rightarrow +\infty} \|\Delta(n)\|_{\mathcal{U} \rightarrow \mathcal{X}} < \frac{1}{\max_{|\zeta|=1} \|\widehat{E}(\zeta)\|_{\mathcal{X} \rightarrow \mathcal{U}}}. \quad (7.2)$$

*Proof.* Define an operator  $E : \mathcal{B}^{1,\gamma} \rightarrow \mathcal{U}$  by  $E\varphi = \sum_{j=0}^{+\infty} E(j)\varphi^{[-j]}$ . Consider system (7.1) as a perturbation of (3.2) with  $K(j) = \theta_{\mathcal{X}}$  for all  $j$ ,  $\mathcal{U}_1 = \mathcal{U}$ ,  $\mathcal{U}_2 = \mathcal{X}$ , and  $D = I_{\mathcal{X}}$ . Then (4.5) implies that (7.1) is UES in  $\mathcal{X}$  w.r.t.  $B^{1,\gamma}$  whenever

$$\sup_{n \geq 0} \|\Delta(n)\|_{\mathcal{U} \rightarrow \mathcal{X}} < \frac{1}{\max_{|\zeta|=1} \|\widehat{E}(\zeta)\|_{\mathcal{X} \rightarrow \mathcal{U}}}.$$

The reference to Lemma 7.1 completes the proof.  $\square$

For operators  $Q(n, j) \in \mathcal{L}(\mathcal{X})$ ,  $n, j \in \mathbb{Z}^+$ , consider the system

$$x(n+1) = \sum_{j=0}^{+\infty} Q(n, j)x(n-j), \quad n \geq 0. \quad (7.3)$$

**Corollary 7.3.** *Let  $\mathcal{X}$  be a Banach space,  $\beta > 0$ , and  $1 \leq p, p' \leq \infty$  be such that  $1/p + 1/p' = 1$ . Assume that for each  $n \geq 0$  the sequence  $\{\|e^{j\beta}Q(n, j)\|_{\mathcal{X} \rightarrow \mathcal{X}}\}_{j=0}^{+\infty}$  belongs to  $\ell^{p'}$ . Then system (7.3) is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}^{p,\beta}$  whenever*

$$\limsup_{n \rightarrow \infty} \sum_{j=0}^{+\infty} \|e^{j\beta}Q(n, j)\|_{\mathcal{X} \rightarrow \mathcal{X}}^{p'} < (1 - e^{-p\beta})^{1/(p-1)} \quad \text{in the case } 1 < p \leq \infty, \quad (7.4)$$

$$\text{and } \limsup_{n \rightarrow \infty} \sup_{j \geq 0} \|e^{j\beta}Q(n, j)\|_{\mathcal{X} \rightarrow \mathcal{X}} < 1 - e^{-\beta} \quad \text{in the case } p = 1, \quad (7.5)$$

where  $e^{-p\beta}$  and  $1/(p-1)$  have to be understood as zero when  $p = \infty$ .

*Proof.* Define operators  $N(n) : \mathcal{B}^{p,\beta} \rightarrow \mathcal{X}$  by  $N(n)\varphi = \sum_{j=0}^{+\infty} Q(n, j)\varphi^{[-j]}$  and consider system (7.3) as an unstructured perturbation of (3.2) with  $K(j) = \theta_{\mathcal{X}}$ ,  $j \geq 0$ . The norm of the unstructured input-state operator  $\Gamma_K$  equals 1 in each of  $\ell^s$ -spaces. By Corollary 5.2, system (7.3) is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}^{p,\beta}$  whenever  $\sup_n \|N(n)\|_{\mathcal{B}^{p,\beta} \rightarrow \mathcal{X}} < (1 - e^{-p\beta})^{1/p}$ . Since

$$\|N(n)\|_{\mathcal{B}^{p,\beta} \rightarrow \mathcal{X}} \leq \left( \sum_{j=0}^{+\infty} \|e^{j\beta}Q(n, j)\|_{\mathcal{X} \rightarrow \mathcal{X}}^{p'} \right)^{1/p'} \quad \text{when } 1 < p \leq \infty, \quad (7.6)$$

$$\text{and } \|N(n)\|_{\mathcal{B}^{1,\beta} \rightarrow \mathcal{X}} \leq \sup_{j \geq 0} \|e^{j\beta}Q(n, j)\|_{\mathcal{X} \rightarrow \mathcal{X}} \quad \text{when } p = 1, \quad (7.7)$$

we see that (7.3) is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}^{p,\beta}$  if

$$\sup_n \sum_{j=0}^{+\infty} \|e^{j\beta}Q(n, j)\|_{\mathcal{X} \rightarrow \mathcal{X}}^{p'} < (1 - e^{-p\beta})^{p'/p} = (1 - e^{-p\beta})^{1/(p-1)} \quad \text{for } 1 < p \leq \infty, \quad (7.8)$$

$$\text{and } \sup_{n, j} \|e^{j\beta}Q(n, j)\|_{\mathcal{X} \rightarrow \mathcal{X}} < 1 - e^{-\beta} \quad \text{for } p = 1. \quad (7.9)$$

Lemma 7.1 completes the proof.  $\square$

Now Proposition 3.2 makes it possible to give sufficient conditions of UE stability w.r.t. the non-fading phase spaces  $\mathcal{B}^{\infty,0}$  and  $\mathcal{B}_0^{\infty,0}$ .

**Corollary 7.4.** *Let  $\mathcal{X}$  be a Banach space,  $0 \leq \gamma < \alpha$ , and  $1 \leq q \leq \infty$ . Assume that for each  $n \geq 0$  the operator  $Q(n) = \sum_{j=0}^{+\infty} Q(n,j)\varphi^{[-j]}$  is bounded in  $\mathcal{B}^{q,\gamma}$  (in  $\mathcal{B}_0^{\infty,\gamma}$ ). Here the convergence of the infinite sum is understood in the sense of the norm topology of  $\mathcal{X}$ .*

*Assume that, for  $n$  large enough, there exist constants  $C(n)$  such that  $Q(n,j) \leq C(n)e^{-j\alpha}$  for all  $j \geq 0$ . Then each of conditions (7.4), (7.5) with arbitrary  $\beta \in (\gamma, \alpha)$  and arbitrary  $p$  in the range  $1 \leq p \leq \infty$  implies the UE stability of system (7.3) in  $\mathcal{X}$  w.r.t.  $\mathcal{B}^{q,\gamma}$  (resp., w.r.t.  $\mathcal{B}_0^{\infty,\gamma}$ ).*

*Proof.* According to the assumptions, it is possible to modify  $Q(n,j)$  for  $0 \leq n \leq n_0 < \infty$  such that the modified system is defined on each of phase spaces  $\mathcal{B}^{p,\beta}$  with  $\beta < \alpha$ . The UE stability of the initial and the modified system in  $\mathcal{X}$  w.r.t.  $\mathcal{B}^{q,\gamma}$  are equivalent due to Lemma 7.1. Applying Corollary 7.3 to the modified system, we see that it is UES in the  $\mathcal{B}^{p,\beta}$  settings. For  $\beta > \gamma$  Proposition 3.2 implies that both the modified and the original system is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}^{q,\gamma}$ . For the case of  $\mathcal{B}_0^{\infty,\gamma}$ , the proof is the same.  $\square$

The condition (7.4) for  $p = \infty$  and  $\beta > 0$  improves the sufficient condition for UE stability in the resolvent matrix sense given by [11, formula (3.1)].

## 7.2. A delayed feedback scheme

Consider another feedback scheme given by Fig.2. Here  $y(n) \in \mathcal{V}_1$  is an output depending now only on

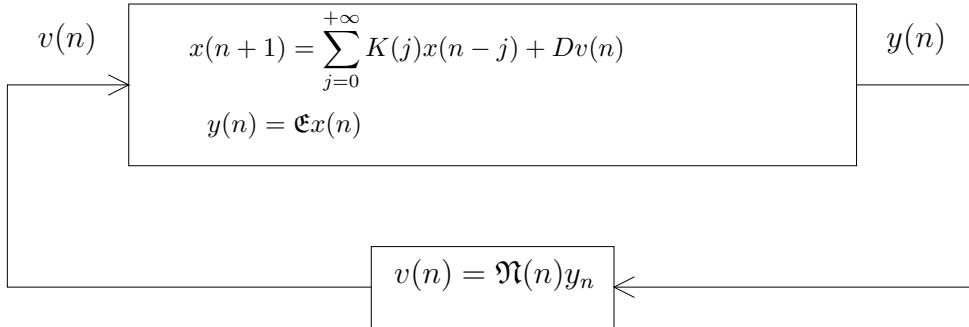


Figure 2: Delayed feedback.

the state  $x(n)$  the system, but the input  $v(n) \in \mathcal{V}_2$  is connected with the output by  $v(n) = \mathfrak{N}(n)y_n$  and so depends on the prehistory of the output. Here the Banach space  $\mathcal{V}_2$  ( $\mathcal{V}_1$ ) is the input (resp., output) space.

In this section, we will use the space  $\mathcal{B}^{p,\gamma}(\mathcal{V}_1)$ , which is defined similar to  $\mathcal{B}^{p,\gamma}$ , but with  $\mathcal{V}_1$  instead of  $\mathcal{X}$  (so that  $\mathcal{B}^{p,\gamma} = \mathcal{B}^{p,\gamma}(\mathcal{X})$ ). Suppose that  $\mathfrak{E} \in \mathcal{L}(\mathcal{X}, \mathcal{V}_1)$  and that the prehistory of the output  $y_n = \{y_n^{[m]}\}_{m=-\infty}^0 := \{y(n+m)\}_{m=-\infty}^0$  belongs to  $\mathcal{B}^{p,\gamma}(\mathcal{V}_1)$ . Then it is natural to assume that *unknown feedback operators*  $\mathfrak{N}(n)$  map  $\mathcal{B}^{p,\gamma}(\mathcal{V}_1)$  to  $\mathcal{V}_2$ . One can define corresponding stability radii similar to that of Section 4.1.

However, we do not want to introduce a new notation because corresponding perturbed systems *can be considered as particular cases* of systems (Sc)-(Snt). For this purpose, consider the diagonal operator

$$M_{\mathfrak{E}} : \mathcal{B}^{p,\gamma} \rightarrow \mathcal{B}^{p,\gamma}(\mathcal{V}_1) \text{ defined by } (M_{\mathfrak{E}}\varphi)^{[m]} = \mathfrak{E}\varphi^{[m]}, \quad m \in \mathbb{Z}^-,$$

and put

$$\mathcal{U}_1 = \mathcal{B}^{p,\gamma}(\mathcal{V}_1), \quad \mathcal{U}_2 = \mathcal{V}_2, \quad \text{and } E = M_{\mathfrak{E}}.$$

Then the following perturbed system can be associated with Fig.2:

$$x(n+1) = \sum_{j=0}^{+\infty} K(j) x(n-j) + D \mathfrak{N}(n) M_{\mathfrak{E}} x_n. \quad (7.10)$$

So  $r_i(D, M_{\mathfrak{E}}; \mathcal{B})$ ,  $i = c, t$ , are the stability radii for the delayed feedback scheme.

*Remark 7.5.* In the case when  $K(j)$  are positive compact operators on a complex Banach lattice  $\mathcal{X}$ , and  $D, E, \mathfrak{N}(n)$  satisfy certain additional assumptions, a radius of asymptotic stability defined similar to  $r_c(D, M_{\mathfrak{E}}; \mathcal{B}_0^{\infty,0})$  was considered in [18, Sect.4].

The input-output operator  $\mathfrak{L}_K : \mathcal{S}_+(\mathcal{U}_2) \rightarrow \mathcal{S}_+(\mathcal{U}_1)$  associated with Fig.2 is defined by  $\mathfrak{L}_K : v(\cdot) \rightarrow y(\cdot)$ , where  $y(n) = \mathfrak{E}x(n)$ ,  $n \geq 0$ , and  $x(\cdot)$  is the solution to the system (4.1).

Note that the operator  $L_K$  associated with (7.10) differs from  $\mathfrak{L}_K$ , though they are obviously connected by

$$(L_K v)(n) = M_{\mathfrak{E}} x_n = \{\dots, \mathfrak{E}x(n-1), \mathfrak{E}x(n)\} = \{\dots, (\mathfrak{L}_K v)(n-1), (\mathfrak{L}_K v)(n)\}.$$

**Corollary 7.6.** *Let  $\gamma > 0$  and  $1 \leq p \leq \infty$ . Let  $\mathcal{B} = \mathcal{B}^{p,\gamma}$  or  $\mathcal{B} = \mathcal{B}_0^{\infty,\gamma}$  (in the latter case  $p$  is assumed to be equal to  $\infty$  and  $M_{\mathfrak{E}}$  is assumed to be restricted to  $\mathcal{B}_0^{\infty,\gamma}$ ). Let (3.2) be UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}$ . Then*

$$\begin{aligned} (1 - e^{-p\gamma})^{1/p} \left( \max_{|\zeta|=1} \|\mathfrak{E}[I_{\mathcal{X}} - \zeta \hat{K}(\zeta)]^{-1} D\|_{\mathcal{V}_2 \rightarrow \mathcal{V}_1} \right)^{-1} &= r_c(D, M_{\mathfrak{E}}; \mathcal{B}) \geq r_t(D, M_{\mathfrak{E}}; \mathcal{B}) \geq \\ &\geq (1 - e^{-p\gamma})^{1/p} \|\mathfrak{L}_K\|_{\ell^p(\mathcal{V}_2) \rightarrow \ell^p(\mathcal{V}_1)}^{-1} > 0, \end{aligned} \quad (7.11)$$

where  $e^{-p\gamma}$  and  $1/p$  have to be understood as zero when  $p = \infty$ .

If  $p = 2$  and  $\mathcal{X}, \mathcal{V}_1, \mathcal{V}_2$  are Hilbert spaces, the equalities hold in (7.11).

*Proof.* It is enough to apply Theorem 4.3 and to perform calculations similar to that of Section 5. The first equality in (7.11) requires additional explanations. The discrete function  $M_{\mathfrak{E}}(\cdot)$  constructed by the operator  $M_{\mathfrak{E}}$  (see Section 2.1) is given by

$$M_{\mathfrak{E}}(n) = M_{\mathfrak{E}} P_{-n}^T = P_{-n}^T \mathfrak{E}.$$

Here we extended the definition of  $P_{-n}^T$  given in Section 2.1 to the space  $\mathcal{B}^{p,\gamma}(\mathcal{V}_1)$ . So

$$\left( \widehat{M}_{\mathfrak{E}}(\zeta) \psi \right)^{[m]} = \zeta^{-m} \mathfrak{E} \psi, \quad m \in \mathbb{Z}^-.$$

When  $|\zeta| = 1$ ,  $v \in \mathcal{V}_2$ , and  $p < \infty$ ,

$$\begin{aligned} |\widehat{M}_{\mathfrak{E}}(\zeta)[I_{\mathcal{X}} - \zeta \hat{K}(\zeta)]^{-1} Dv|_{\mathcal{U}_1}^p &= |\widehat{M}_{\mathfrak{E}}(\zeta)[I_{\mathcal{X}} - \zeta \hat{K}(\zeta)]^{-1} Dv|_{\mathcal{B}^{p,\gamma}(\mathcal{V}_1)}^p = \\ &= \sum_{m=-\infty}^0 e^{pm\gamma} |\mathfrak{E}[I_{\mathcal{X}} - \zeta \hat{K}(\zeta)]^{-1} Dv|_{\mathcal{V}_1}^p = (1 - e^{-p\gamma})^{-1} |\mathfrak{E}[I_{\mathcal{X}} - \zeta \hat{K}(\zeta)]^{-1} Dv|_{\mathcal{V}_1}^p. \end{aligned}$$

This gives the desired equality (with standard changes for  $p = \infty$ ).

To get the last inequality in (7.11), we use the formula

$$\|L_K\|_{\ell^p(\mathcal{V}_2) \rightarrow \ell^p(\mathcal{B}^{p,\gamma}(\mathcal{V}_1))} = (1 - e^{-p\gamma})^{-1/p} \|\mathfrak{L}_K\|_{\ell^p(\mathcal{V}_2) \rightarrow \ell^p(\mathcal{V}_1)},$$

which can be obtained in the same way as (5.4). □

**Corollary 7.7.** *If  $\mathfrak{E} \neq 0_{\mathcal{X} \rightarrow \mathcal{V}_1}$  and  $D \neq 0_{\mathcal{V}_2 \rightarrow \mathcal{X}}$ , then  $r_i(D, M_{\mathfrak{E}}; \mathcal{B}_0^{\infty,0}) = 0$ ,  $i = c, t$  (here  $M_{\mathfrak{E}}$  is assumed to be restricted to  $\mathcal{B}_0^{\infty,0}$ ).*

*Proof.* In this case,  $\mathcal{B} = \mathcal{B}_0^{\infty,0}$ . The function  $M_{\mathfrak{E}}(\cdot)$  (which corresponds to  $E(\cdot)$  of Proposition 4.5) does not decay exponentially since  $\|M_{\mathfrak{E}}(n)\|_{\mathcal{X} \rightarrow \mathcal{U}_1} = \|P_{-n}^T \mathfrak{E}\|_{\mathcal{X} \rightarrow \mathcal{B}_0^{\infty,0}(\mathcal{V}_1)} = \|\mathfrak{E}\|_{\mathcal{X} \rightarrow \mathcal{V}_1}$  is a positive constant. Proposition 4.5 completes the proof. □

### 7.3. An example of a perturbed non-positive system

Take  $\mathcal{X} = \mathbb{C}$  and consider the systems

$$x(n+1) = -\sum_{j=0}^{\infty} 2^{-j} x(n-j) + \sum_{j=0}^{+\infty} \Delta(n, j) x(n-j), \quad n \geq 0, \quad (7.12)$$

with uncertain complex coefficients  $\Delta(n, j) \in \mathbb{C}$ ,  $n, j \in \mathbb{Z}^+$ . The problem is to find conditions on  $\Delta(n, j)$  that ensure the UE stability of these systems in  $\mathcal{X}$  with respect to a certain phase space (by Remark 2.5, such conditions guarantee also the UE stability in the resolvent matrix sense).

We consider systems (7.12) as perturbations of the convolution system

$$x(n+1) = -\sum_{j=0}^{\infty} 2^{-j} x(n-j). \quad (7.13)$$

First, consider stability properties of system (7.13). It is a system of the type (3.2) with  $K(j) = -2^{-j}$ . The Z-transform of  $K(\cdot)$  equals  $\widehat{K}(\zeta) = \frac{2}{\zeta-2}$ . The radius of convergence of  $\widehat{K}(\zeta)$  equals  $R[\widehat{K}] = 2$ . System (7.13) is defined on the phase spaces

$$\mathcal{B}^{p,\gamma} \text{ for all } -\infty < \gamma < \ln 2, \quad 1 \leq p \leq \infty, \text{ and also on } \mathcal{B}^{1, \ln 2}. \quad (7.14)$$

Recall that  $X_K(\cdot)$  is the convolution kernel corresponding to the unstructured input-state operator  $\Gamma_K$ , see (3.7), and that  $X_K(\cdot)$  is connected with the resolvent matrix  $X_K(\cdot, \cdot)$  of (7.13) by  $X_K(n, j) = X_K(n-j)$ . According to Lemma 3.7, the Z-transform of the function  $X_K(\cdot)$  equals

$$\widehat{X}_K(\zeta) = [1 - \zeta \widehat{K}(\zeta)]^{-1} = \frac{2 - \zeta}{2 + \zeta}. \quad (7.15)$$

Recovering the function  $X_K(\cdot)$  from its Z-transform, one gets  $X(0) = 1$  and  $X(j) = -(-\frac{1}{2})^{j-1}$  for  $j \in \mathbb{N}$ . By (3.7), the explicit form of the the unstructured input-state operator  $\Gamma_K$  is

$$(\Gamma_K f)(n) = f(n-1) - \sum_{j=2}^n \left(-\frac{1}{2}\right)^{j-2} f(n-j), \quad (\Gamma_K f)(0) = 0. \quad (7.16)$$

We see that  $X_K(\cdot)$  decays exponentially. In other words, *system (7.13) is UES in the resolvent matrix sense*. By Theorem 3.1 and Proposition 3.2, *system (7.13) is UES in  $\mathcal{X}$  w.r.t. each of the phase spaces of (7.14)*. For  $\gamma < \ln 2$ , *system (7.13) is also UES in  $\mathcal{X}$  w.r.t. the spaces  $\mathcal{B}_0^{\infty, \gamma}$* , which are isometrically embedded in  $\mathcal{B}^{\infty, \gamma}$ .

Let us study stability radii of (7.13) under unstructured perturbations in a phase space  $\mathcal{B} = \mathcal{B}^{p,\gamma}$  assuming that either  $0 < \gamma < \ln 2$ ,  $1 \leq p \leq \infty$ , or  $\gamma = \ln 2$ ,  $p = 1$ . We want to use the settings of Section 5 to calculate (or estimate) the stability radii  $r_c(K; \mathcal{B})$  and  $r_t(K; \mathcal{B})$ .

Clearly,

$$\max_{|\zeta|=1} |[1 - \zeta \widehat{K}(\zeta)]^{-1}| = \max_{|\zeta|=1} \left| \frac{2 - \zeta}{2 + \zeta} \right| = 3.$$

By Corollary 5.2,

$$r_c(K; \mathcal{B}^{p,\gamma}) = \frac{(1 - e^{-p\gamma})^{1/p}}{3}.$$

Time-varying stability radii  $r_t$  can be easily calculated when  $p = 1, 2, \infty$ :

$$\frac{(1 - e^{-p\gamma})^{1/p}}{3} = r_t(K; \mathcal{B}^{p,\gamma}) = (1 - e^{-p\gamma})^{1/p} \|\Gamma_K\|_{\ell^p(\mathcal{X}) \rightarrow \ell^p(\mathcal{X})}^{-1}, \quad p = 1, 2, \infty. \quad (7.17)$$



Indeed, for  $p = 2$  this equality is provided immediately by Corollary 5.2. When  $p = 1$  or  $p = \infty$ , the norms of the unstructured input-state operator can be calculated via (7.16):

$$\|\Gamma_K\|_{\ell^1(\mathcal{X}) \rightarrow \ell^1(\mathcal{X})} = \|\Gamma_K\|_{\ell^\infty(\mathcal{X}) \rightarrow \ell^\infty(\mathcal{X})} = 3.$$

(The supremum of norms of  $\Gamma_K f$  over the corresponding unit balls are archived for suitable  $f(\cdot)$  with alternating signs of  $f(n)$ .) Hence,  $\max_{|\zeta|=1} |[1 - \zeta \hat{K}(\zeta)]^{-1}| = \|\Gamma_K\|_{\ell^p(\mathcal{X}) \rightarrow \ell^p(\mathcal{X})}$ , and therefore, (5.2) holds with equalities. This proves (7.17).

Finally, we derive stability conditions for (7.12) in terms of coefficients using the obtained stability radii. To write system (7.12) in the form (5.1), we define the operators (actually, the functionals)  $N(n) : \mathcal{B}^{p,\gamma} \rightarrow \mathbb{C}$  by  $N(n)\varphi = \sum_{j=0}^{+\infty} \Delta(n, j) \varphi^{[-j]}$ . Then,

$$\|N(n)\|_{\mathcal{B}^{p,\gamma} \rightarrow \mathbb{C}} = \left( \sum_{j=0}^{+\infty} |\Delta(n, j) e^{j\gamma}|^{p'} \right)^{1/p'} \quad \text{when } 1 < p \leq \infty, \quad (7.18)$$

$$\text{and } \|N(n)\|_{\mathcal{B}^{1,\gamma} \rightarrow \mathbb{C}} = \sup_{j \geq 0} |\Delta(n, j) e^{j\gamma}| \quad \text{when } p = 1, \quad (7.19)$$

where  $p'$  is the Hölder conjugate of  $p$ ,  $1/p' + 1/p = 1$ .

Combining the definition of  $r_t(K; \mathcal{B}^{p,\beta})$  with (7.17), we see that (7.12) is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}^{p,\beta}$  in each of the following cases:

$p = 1$ ,  $0 < \beta \leq \ln 2$ , and

$$\sup_{j \geq 0} |\Delta(n, j) e^{j\beta}| < \frac{1 - e^{-\beta}}{3} \quad \text{for all } n \geq 0; \quad (N1)$$

$p = 2$ ,  $0 < \beta < \ln 2$ , and

$$\sum_{j=0}^{+\infty} |\Delta(n, j) e^{j\beta}|^2 < \frac{1 - e^{-2\beta}}{9} \quad \text{for all } n \geq 0; \quad (N2)$$

$p = \infty$ ,  $0 < \beta < \ln 2$ , and

$$\sum_{j=0}^{+\infty} |\Delta(n, j) e^{j\beta}| < \frac{1}{3} \quad \text{for all } n \geq 0. \quad (N\infty)$$

The continuous embedding  $\mathcal{B}^{1,\delta} \subset \mathcal{B}^{2,\delta} \subset \mathcal{B}^{\infty,\delta}$  and Proposition 3.2 imply also that, in the case  $p = 1$ ,  $0 < \beta < \ln 2$ , (7.12) is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}^{1,\beta}$  whenever any of the conditions (N2) or (N $\infty$ ) is satisfied.

Note that conditions (N1), (N2), and (N $\infty$ ) are independent, i.e., none of them implies another one.

Similarly, in the case  $p = 2$ ,  $0 < \beta < \ln 2$ , (7.12) is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}^{2,\beta}$  whenever (N $\infty$ ) is satisfied.

The unstructured stability radii corresponding to  $\mathcal{B} = \mathcal{B}^{\infty,0}$  and  $\mathcal{B} = \mathcal{B}_0^{\infty,0}$  do not produce stability tests since these radii are equal to 0, see Corollary 5.1. However, the continuous embedding argument allows one to obtain sufficient conditions of UE stability in  $\mathcal{X}$  w.r.t.  $\mathcal{B}^{\infty,0}$  and  $\mathcal{B}_0^{\infty,0}$ , as well as w.r.t.  $\mathcal{B}^{p,\gamma}$  with  $p \neq 1, 2, \infty$ . In fact, embedding (2.4), Proposition 3.2, and the above results yield the following conditions (since the produced conditions for the phase spaces  $\mathcal{B}^{\infty,0}$  and  $\mathcal{B}_0^{\infty,0}$  coincide, below we give only  $\mathcal{B}^{\infty,0}$  version).

**Proposition 7.8.** *Let  $0 \leq \gamma < \ln 2$ . System (7.12) is UES in  $\mathcal{X}$  w.r.t.  $\mathcal{B}^{p,\gamma}$  if the condition (N1) is fulfilled for a certain  $\beta \in (\gamma, \ln 2]$  or if any of the conditions (N2), (N $\infty$ ) is fulfilled for a certain  $\beta \in (\gamma, \ln 2)$ .*

These scales of stability tests have the following additional properties:

- (i) as before, none of the above conditions imply another one (even produced by a different  $\beta$ ),
- (ii) the constants in the right sides of (N1), (N2), and (N $\infty$ ) are sharp, more precisely, for each of the conditions (N1), (N2), and (N $\infty$ ), there exist  $\Delta(n, j)$  such that the equality holds in the corresponding formula, but (7.12) is not UES in  $\mathcal{X}$  w.r.t. any of phase spaces  $\mathcal{B}^{p,\gamma}$ .

- (iii) using Lemma 7.1, the requirement 'for all  $n \geq 0$ ' in (N1), (N2), and (N $\infty$ ) can be weakened to 'for all  $n$  large enough'.

Statement (i) can be easily seen by direct examination.

Let us prove (ii) for the case of (N $\infty$ ). Taking  $\Delta(n, 0) = -1/3$  for all  $n \geq 0$ , and  $\Delta(n, j) = 0$  for all  $j \geq 1$  and  $n \geq 0$ , we see by straightforward calculations that the equality holds in (N $\infty$ ), that the convolution (time-invariant) system (7.12) is defined for all phase spaces of (7.14), and that for system (7.12) the condition (ii) of Theorem 3.1 is not valid when  $\zeta = -1$ . Hence, (7.12) is not UES in the resolvent matrix sense. The equality holds in (N1) if  $\Delta(n, j) = -\frac{1-e^{-\beta}}{3}(-1)^j e^{-\beta j}$  for all  $n$ . Though the corresponding convolution system (7.12) is defined on  $\mathcal{B}^{1,\beta}$  and all the embedded phase spaces, it is not UES in the resolvent matrix sense. Indeed, condition (ii) of Theorem 3.1 is not valid again for  $\zeta = -1$ . Taking  $\Delta(n, j) = -\frac{1-e^{-2\beta}}{3}(-1)^j e^{-2\beta j}$ , we see that the equality holds in (N2), but the system it is not UES in the resolvent matrix sense by the same reason as before.

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